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DYNAMICS.

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A TREATISE

ON

DYNAMICS.

BY

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T

PREFACE.

THE object of the present Treatise is to introduce the mathematical student to some of the earlier and easier branches of Kinematics and Kinetics.

In the chapter allotted to Kinematics, I have deduced the expressions for velocities and accelerations, as far as possible, from the definitions and axioms of the subject.

In the applications to Kinetics, or, in other words, in the combination of these expressions with the Laws of Motion, for the determination of the motion of a particle or of a system, I have adopted the same plan of operations.

I have assumed, and made free use of, the methods of Analysis, for the performance and simplification of the requisite calculations.

The methods employed, and the order of thought which is followed, are those which during my experience as a teacher I have found to be most effective in the elucidation and development of the ideas of Kinetics.

The majority of students do not easily or rapidly absorb general ideas, and they are most effectively assisted by the gradual development of a subject through simple cases, and illustrative examples.

With this view I have endeavoured to explain the application of the Laws of Motion to the determination of the motion of a particle and of systems of particles, commencing with easy cases, and leading up to a few of the interesting and important cases of the motion of a body in space.

My especial object has been to illustrate the direct application of the Laws of Motion, and thereby to produce a treatise of an elementary character, but of Educational utility to the student who is commencing the study of theoretical Kinetics.

I am very much indebted to Professor W. H. H. Hudson, of King's College, London, for useful suggestions and criticisms, and to Mr G. B. Mathews, Professor of Mathematics at the North Wales University, and Mr J. Brill, B.A. of St John's College, Cambridge, for their kind assistance in the correction of manuscripts and proof sheets.

I venture to hope that the explanations and illustrations of the text, and the numerous examples appended to the several chapters, will be of assistance to the student in mastering the elementary ideas of the subject, and pave the way for the consideration of the higher branches and the more difficult problems of the great science of Dynamics.

W. H. BESANT.

September 16, 1884.

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DYNAMICS.

CHAPTER I.

1. THE problems usually discussed under this head are those which relate to the geometrical connections between given motions, or given kinds of motion, and those which relate to the action of forces, and the motions and changes of motion produced by forces.

The former belong to pure science, and deal with the geometry of motion, a branch of mathematics to which the name Kinematics was applied by Ampère.

Strictly speaking the word Dynamics includes Statics, the discussion of the equilibrium or balancing of forces, and Kinetics, the discussion of the effects of forces on the motion of bodies. Mechanism belongs to Kinematics, including such problems as result from considering trains of wheelwork or any connected machinery.

To Kinetics belong the consideration of the forces setting such machinery in motion, or keeping it in motion, the problems of Physical Astronomy, and others of important practical application.

We shall commence by a development of the formulæ of Kinematics, and afterwards proceed to consider the application of the formulæ, and of the Laws of Motion, or Laws of Force, to the determination of the motion of a particle, and of a system of particles, produced by the action of given forces, or, conversely, of the forces required to produce given motions. The idea of a particle, or of a material point, capable of being set in motion, or of having its motion affected, by the action of force, is a mathematical abstraction leading to the simplest forms of Kinetics. The determination of the motions of the bodies constituting the Solar System belongs to this class in virtue of the facts that the Planetary Bodies are nearly spherical in form, and that their dimensions are very small in comparison with their distances from each other and from the Sun.

Moreover the mathematical idea of a solid body is that of a system of particles, and the discussion of the motion of a single particle therefore naturally precedes the discussion of the motion of a body or system of particles.

It will be seen that Newton's Laws of Motion connect the action of a force on a particle with the accelerations produced, and lead to the formation of differential equations, the integrations of which give the solution of the problem of determining the motion.

It will appear further that Newton's Laws are sufficient for the determination of the motion of a system of particles or bodies, whether rigidly connected or not, and lead, in a similar manner, to systems of differential equations containing, in their solution, the motions of the body, or of the various bodies of the system.

CHAPTER II.

DIFFERENTIAL EQUATIONS.

2. THERE are certain differential equations which occur so frequently in the discussion of questions in Kinetics, that we think it worth while, for convenience of reference, to give a brief solution of them.

(1) The equation, $\frac{dy}{dx}+yf'(x)=F(x)$, is at once solved by the integrating factor

$$\epsilon^{f(x)},$$

leading to $$y\epsilon^{f(x)}=\int F(x)\,\epsilon^{f(x)}\,dx+C.$$

For example, if $\frac{dy}{dx}+\frac{y}{\sin x}=\tan\frac{x}{2}$, the factor is $\tan\frac{x}{2}$, and therefore

$$y\tan\frac{x}{2}=2\tan\frac{x}{2}-x+C.$$

(2) $$\frac{d^2y}{dx^2}+n^2y=0.$$

Multiply by $2\frac{dy}{dx}$ and integrate, then

$$\left(\frac{dy}{dx}\right)^2=n^2(c^2-y^2) \text{ or } \frac{dx}{dy}=\pm\frac{1}{n}\frac{1}{\sqrt{c^2-y^2}};$$

$$\therefore y=c\cos(nx+\alpha),$$

or $$y=A\cos nx+B\sin nx.$$

(3)
$$\frac{d^2y}{dx^2} - n^2y = 0.$$

As before,

$$\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + c^2) \text{ or } \frac{dx}{dy} = \pm\frac{1}{n}\frac{1}{\sqrt{y^2 + c^2}};$$

$$\therefore\ nx = \log\frac{y + \sqrt{y^2 + c^2}}{C},$$

from which we obtain

$$y = A\epsilon^{nx} + B\epsilon^{-nx}.$$

The equation $\frac{d^2y}{dx^2} \pm n^2y = ax + b$ is reduced to one of the two preceding by assuming

$$y \mp \frac{ax + b}{n^2} = z.$$

(4) If
$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 f(y) = F(y), \text{ let } \frac{dy}{dx} = z,$$

then
$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = z\frac{dz}{dy} = \tfrac{1}{2}\cdot\frac{d\,.\,z^2}{dy},$$

and the equation becomes, putting v for z^2,

$$\frac{dv}{dy} + 2f(y)\,v = 2F(y),$$

which is of the form (1).

(5)
$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \epsilon^{nx} + x^r.$$

The solution of this equation is effected by the calculus of operations, leading to

$$y = \frac{\epsilon^{nx}}{n^2 + an + b} + \frac{1}{b}\left(1 + \frac{a}{b}D + \frac{D^2}{b}\right)^{-1} x^r,$$

D representing the operation $\frac{d}{dx}$, and the expression affecting x^r being expanded in ascending powers of D.

The complementary function $A\epsilon^{\alpha x} + B\epsilon^{\beta x}$ must be added, α and β being the roots of the equation

$$m^2 + am + b = 0.$$

If the roots of this equation are imaginary and of the form $\alpha \pm \beta\sqrt{\overline{1}}$, the complementary function takes the form

$$\epsilon^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

(6) $$\frac{d^2y}{dx^2} - n^2y = \cos rx.$$

The calculus of operations, or the variation of parameters, gives as the integral,

$$y = -\frac{\cos rx}{r^2 + n^2} + A\epsilon^{nx} + B\epsilon^{-nx}.$$

(7) $$\frac{d^2y}{dx^2} + n^2y = \cos rx.$$

The solution is

$$y = \frac{\cos rx}{n^2 - r^2} + A\cos nx + B\sin nx.$$

If $n = r$, then, writing the solution in the form

$$y = \frac{\cos rx - \cos nx}{n^2 - r^2} + A\cos nx + B\sin nx,$$

and finding the limiting value of the fraction when $n = r$, we obtain

$$y = \frac{x}{2n}\sin nx + A\cos nx + B\sin nx.$$

(8) $$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \cos rx.$$

This leads to

$$y = \frac{D^2 - aD + b}{(D^2 + b)^2 - a^2D^2}\cos rx = (D^2 - aD + b)\frac{\cos rx}{(b - r^2)^2 + a^2r^2},$$

the complementary function of course being added;

$$\therefore y = \frac{(b - r^2)\cos rx + ar\sin rx}{(b - r^2)^2 + a^2r^2} + A\epsilon^{\alpha x} + B\epsilon^{\beta x},$$

α and β being the roots of the equation,

$$z^2 + az + b = 0.$$

CHAPTER III.

KINEMATICS.

3. *Definition.* The velocity of a moving point, when uniform, is measured by the number of units of length passed over in the unit of time.

If the velocity be not uniform, it is measured at any instant by the space which would be passed over in the unit of time if the velocity were to remain the same as it is at that instant.

To express this idea mathematically, let s be the space, that is, the number of units of length passed over by the moving point in the time t, and $s + \delta s$ the space passed over in the time $t + \delta t$, so that δs is the space passed over in the time δt, and, if δt be so small that the velocity is not sensibly changed in the time δt, the limiting value of the expression $\frac{\delta s}{\delta t}$, that is $\frac{ds}{dt}$ is the measure of the velocity.

Or we may argue as follows, If v be the velocity at the time t and $v + \delta v$ at the time $t + \delta t$, then δs lies between $v\delta t$ and $(v + \delta v)\,\delta t$, and therefore ultimately,

$$v = \frac{ds}{dt}.$$

4. If the velocity of a moving point be variable, it is said to have positive acceleration, if the velocity be increasing, and negative acceleration, or retardation, if the velocity be decreasing.

If the rate of increase of the velocity be uniform the acceleration is measured by the increase of the velocity in the unit of time, and, if variable, it is measured at any instant by what would be the increase of velocity in the unit of time if the rate of increase were to remain what it is at the instant in question.

Mathematically, if v be the velocity at the time t, and $v + \delta v$ at the time $t + \delta t$, the measure of the acceleration, in the direction of motion, is $\frac{dv}{dt}$.

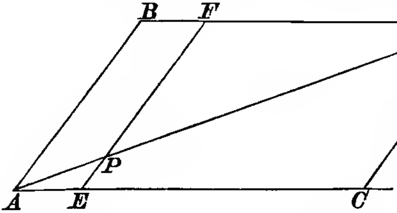
Or, if f be the acceleration at the time t, and $f + \delta f$ at the time $t + \delta t$, δv lies between $f \delta t$ and $(f + \delta f)\, \delta t$, and therefore ultimately,

$$f = \frac{dv}{dt}.$$

5. *The composition and decomposition of velocities and accelerations.*

Supposing a moving point to possess, or to have impressed upon it, two velocities in different directions, there arises the question; what is the resulting motion?

To solve this kinematical question, we must invent machinery to represent coexistent velocities.



Imagine then a point to move uniformly along a straight line while the line is carried parallel to itself at a uniform rate. The point will then have two coexistent velocities.

Let EF be the moving line, and, while the point moves from E to F, let the line move from AB to CD. Then PE is to EA as the velocity of the point along the line is to the velocity of the line, that is, as CD to AC;

Therefore APD is a straight line, and the point P actually moves from A to D.

AD therefore represents the resultant velocity in magnitude and direction.

This proposition is called the *Parallelogram of velocities.*

In the same way, if a point have two coexistent accelerations, taking AB and AC to represent the velocities added per unit of time, or which would be added per unit of time, due to the accelerations at the instant in question, it follows that AD represents the resultant velocity superposed, or which would be superposed, per unit of time.

This is the *Parallelogram of accelerations.*

Conversely, any velocity or acceleration, represented by a line AD, can be decomposed into two velocities or accelerations, AB, AC, in any assigned directions.

6. *Change of units in the measures of velocities and accelerations.*

If v be the measure of a velocity, the meaning is that v units of length are passed over in the unit of time.

If a feet and t seconds be the units, and if v' be the measure of the same velocity when a' feet and t' seconds are units, it follows that

$$\frac{av}{t} = \frac{a'v'}{t'},$$

for each expression represents the velocity in feet per second.

If f be the measure of an acceleration when a feet and t seconds are units, the meaning is that

the velocity per t seconds added in t seconds $= fa$ in feet;

$\therefore$ velocity per second added in t seconds $= \dfrac{fa}{t}$;

$\therefore$ velocity per second added in one second $= \dfrac{fa}{t^2}$.

If f' be the measure of the same acceleration when a' feet and t' seconds are units, it follows that $\dfrac{f'a'}{t'^2}$ is the measure of the same acceleration referred to a foot and a second, and

$$\therefore \frac{fa}{t^2} = \frac{f'a'}{t'^2}.$$

7. *Angular velocity and angular acceleration.*

If a straight line turn round in a plane it is said to have angular velocity, and if this angular velocity be variable it is said to have angular acceleration.

If θ be the inclination, at the time t, of the moving line to any fixed line in the plane, then, exactly as in Articles (3) and (4) the angular velocity is $\frac{d\theta}{dt}$ or $\dot{\theta}$, and the angular acceleration is $\frac{d^2\theta}{dt^2}$, or $\ddot{\theta}$.

It must be observed that this is quite independent of any motion of translation which the line may have, and simply measures the rate of turning round.

When we speak of the angular velocity of a point P, moving in a plane, about a fixed point O in the plane, we really mean the angular velocity of the straight line OP.

If the velocity and direction of motion of P be given, a simple expression can be obtained for its angular velocity about a fixed point O.

For if $OP = r$, and if p be the perpendicular from O on the direction of motion of the point, and v its velocity, the angular velocity is equal to the resolved part of the velocity perpendicular to OP divided by OP, and therefore

$$= v\frac{p}{r} \div r = \frac{pv}{r^2}.$$

8. *Expressions for accelerations.*

If x, y, z be the co-ordinates of a point, referred to a system of fixed axes at right angles to each other, x, y, and z are the distances of the point from the planes of yz, zx, and xy.

The velocity parallel to x is the rate of increase of the distance from the plane yz, and, as in Art. (3), is represented by $\frac{dx}{dt}$ or by $\dot{x}$, employing fluxional notation.

And, similarly $\frac{dy}{dt}$ and $\frac{dz}{dt}$, or $\dot{y}$ and $\dot{z}$, are the velocities parallel to y and z.

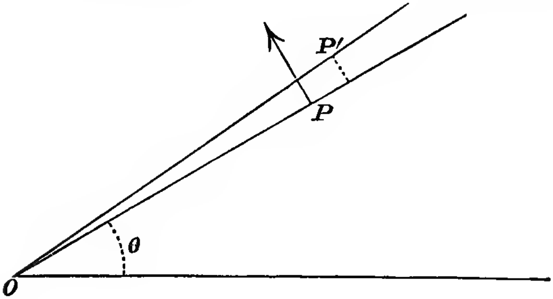
If u, v, w be these velocities, the accelerations parallel to the axes are, by the same reasoning as in Art. (4), $\frac{du}{dt}$, $\frac{dv}{dt}$, and $\frac{dw}{dt}$, or $\dot{u}$, $\dot{v}$, and $\dot{w}$; that is, are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$, or $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$.

We shall in all cases limit the use of the symbols $\dot{x}$, $\ddot{x}$ to the case in which the time is the independent variable.

9. It must be carefully observed that the velocity of a point in any direction is the rate of change of the distance in that direction, and is equal to the limit of the change of distance divided by the change of time, when that change is indefinitely small.

And similarly, the acceleration in any direction is the rate of change of the velocity in that direction, and is equal to the limit of the change of velocity divided by the change of time, when that change is indefinitely small.

10. *Radial and transversal velocities and accelerations.*



Let r, θ be the polar co-ordinates of a moving point, and u, v the radial and transversal velocities, that is, the velocities in direction of OP and perpendicular to OP.

P being the position of the point at the time t, and P' at the time $t + \delta t$, and if $OP' = r + \delta r$,

$$u = \text{limit of } \frac{OP' \cos\delta\theta - OP}{\delta t} = \frac{dr}{dt} = \dot{r},$$

and $$v = \text{limit of } \frac{OP' \sin\delta\theta}{\delta t} = \frac{rd\theta}{dt} = r\dot{\theta}.$$

If $u + \delta u$, $v + \delta v$, be the velocities at P' in direction of and perpendicular to OP', acceleration in direction OP

$$= \text{limit of } \frac{(u + \delta u)\cos d\theta - (v + \delta v)\sin d\theta - u}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt}$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2,$$

acceleration perpendicular to OP

$$= \text{limit of} \frac{(v + \delta v)\cos d\theta + (u + \delta u)\sin d\theta - v}{\delta t}$$

$$= \frac{dv}{dt} + u\frac{d\theta}{dt} = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}$$

$$= r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

It should be noticed that, if $r = 0$, that is, if the moving point is passing through the origin, these expressions are

$$\ddot{r} \text{ and } 2\dot{r}\dot{\theta}.$$

11. The expressions for radial and transversal accelerations may otherwise be obtained in the following manner.

The component accelerations parallel to x and y being $\ddot{x}$ and $\ddot{y}$, the radial and transversal accelerations are respectively

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta, \text{ and } \ddot{y}\cos\theta - \ddot{x}\sin\theta.$$

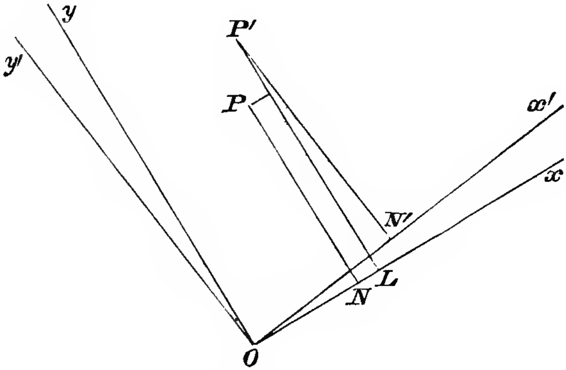
Putting $r\cos\theta$ for x, and $r\sin\theta$ for y, these expressions become

$$\ddot{r} - r\dot{\theta}^2, \text{ and } r\ddot{\theta} + 2\dot{r}\dot{\theta}.$$

12. *Case of uniform motion in a circle.*

If r and $\dot{\theta}$ are both constant, and if $\dot{\theta} = \omega$, the transversal acceleration vanishes, and the radial acceleration $= -\omega^2 r$; that is, the resultant acceleration is directed to the centre of the circle and is equal to the radius multiplied by the square of the angular velocity.

13. *Velocities and accelerations referred to two axes, at right angles to each other, moving in a plane about the origin.*



If Ox, Oy be the positions of the axes at the time t, Ox', Oy' at the time $t + \delta t$, $xOx' = \delta\theta$, and u, v, velocities parallel to x and y at the time t,

$$u = \text{Lt}\,\frac{NL}{dt} = \text{Lt}\,\frac{(x+\delta x)\cos\delta\theta - (y+\delta y)\sin\delta\theta - x}{\delta t} = \frac{dx}{dt} - y\frac{d\theta}{dt},$$

$$v = \text{Lt}\,\frac{(x+\delta x)\sin\delta\theta + (y+\delta y)\cos\delta\theta - y}{\delta t} = \frac{dy}{dt} + x\frac{d\theta}{dt}.$$

If $u + \delta u$, $v + \delta v$ be the velocities of P' parallel to x' and y',

acceleration parallel to Ox

$$= \text{Lt} \frac{(u + \delta u)\cos\delta\theta - (v + \delta v)\sin\delta\theta - u}{dt},$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt} = \frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2 - y\frac{d^2\theta}{dt^2} - 2\frac{dy}{dt}\frac{d\theta}{dt},$$

and acceleration parallel to Oy

$$= \text{Lt} \frac{(u + \delta u)\sin\delta\theta + (v + \delta v)\cos\delta\theta - v}{\delta t},$$

$$= \frac{dv}{dt} + u\frac{d\theta}{dt} = \frac{d^2y}{dt^2} - y\left(\frac{d\theta}{dt}\right)^2 + x\frac{d^2\theta}{dt^2} + 2\frac{dx}{dt}\frac{d\theta}{dt}.$$

In fluxional notation, the velocities are

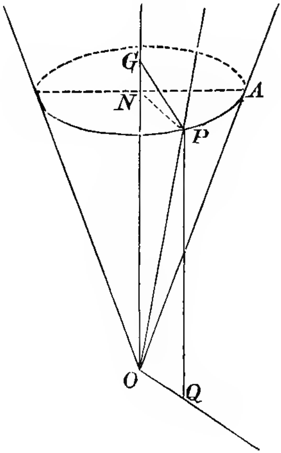
$$\dot{x} - y\dot{\theta}, \text{ and } \dot{y} + x\dot{\theta};$$

and the accelerations are

$$\dot{u} - v\dot{\theta} \text{ and } \dot{v} + u\dot{\theta},$$

or $$\ddot{x} - y\ddot{\theta} - x\dot{\theta}^2 - 2\dot{y}\dot{\theta}, \text{ and } \ddot{y} + x\ddot{\theta} - y\dot{\theta}^2 + 2\dot{x}\dot{\theta}.$$

14. *Case of a point moving on the surface of a right circular cone.*



PN being perpendicular to the axis, let $OP = r$, and the angle between the moving plane NPO and a fixed plane $AON = \theta$, and let Q be the projection of P on a plane through O perpendicular to the axis of the cone.

Acceleration parallel to $ON = \frac{d^2 . ON}{dt^2} = \frac{d^2 r}{dt^2} \cos \alpha \ldots..(1)$,

acceleration in direction NP = that of Q in direction OQ,

$$= \frac{d^2 . OQ}{dt} - OQ \left(\frac{d\theta}{dt}\right)^2, \text{ by Art. (10)} \ldots\ldots\ldots.(2),$$

$$= \frac{d^2 r}{dt^2} \sin \alpha - r \sin \alpha \left(\frac{d\theta}{dt}\right)^2,$$

and perpendicular to the plane NPO

$$= \frac{1}{r \sin \alpha} \frac{d}{dt} \left(r^2 \sin^2 \alpha \frac{d\theta}{dt}\right) = \frac{\sin \alpha}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt}\right) \ldots..(3).$$

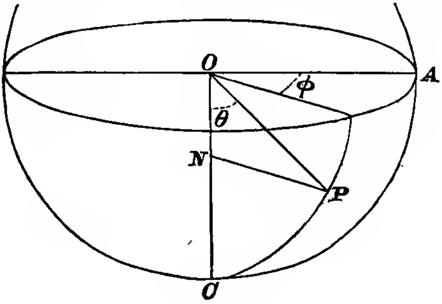
Multiplying (1) by $\cos \alpha$, (2) by $\sin \alpha$, and adding, we find that the acceleration in the direction OP

$$= \frac{d^2 r}{dt^2} - r \sin^2 \alpha \left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r \sin^2 \alpha \dot{\theta}^2.$$

Multiplying (1) by $\sin \alpha$, (2) by $\cos \alpha$, and subtracting, the acceleration in the direction of the normal PG to the surface

$$= r \sin \alpha \cos \alpha \left(\frac{d\theta}{dt}\right)^2 = r \sin \alpha \cos \alpha \dot{\theta}^2.$$

15. *Case of a point moving on the surface of a sphere.*



OAC being a fixed plane, and *OC* a fixed radius, and *PN* perpendicular to *OC*, take ϕ as the angle between the planes *OAC* and *OPC*, and θ as the angle *COP*.

The accelerations in the directions, *ON*, *NP*, and perpendicular to the plane *OPC*, are respectively

$$\frac{d^2}{dt^2}(a\cos\theta),\quad \frac{d^2}{dt^2}(a\sin\theta) - a\sin\theta\left(\frac{d\phi}{dt}\right)^2,$$

and

$$\frac{1}{a\sin\theta}\frac{d}{dt}\left(a^2\sin^2\theta\frac{d\phi}{dt}\right).$$

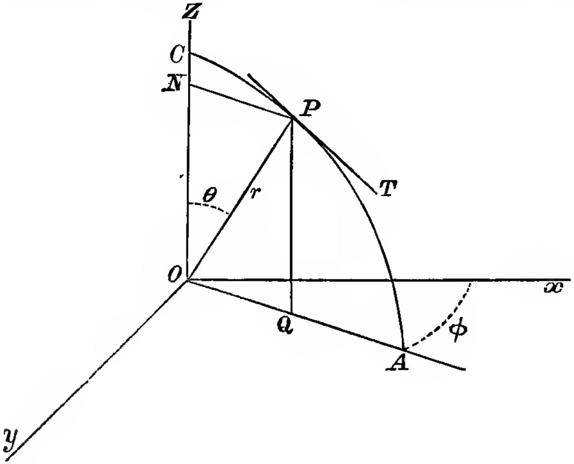
Multiplying the second of these by $\cos\theta$, and the first by $\sin\theta$, the difference of the products is the acceleration in direction of the tangent at *P* to the meridian curve *CP*, and is equal to

$$a\ddot{\theta} - a\sin\theta\cos\theta\,.\,\dot{\phi}^2.$$

Similarly the acceleration in the direction *PO*

$$= a\dot{\theta}^2 + a\sin^2\theta\,.\,\dot{\phi}^2.$$

16. In the general case in which the position of a moving point at any time is defined by the polar co-ordinates r, θ, ϕ, the accelerations of *P* in the directions perpendicular



to Oz in the plane CPA, and perpendicular to that plane are the same as the accelerations of Q, and are therefore

$$\frac{d^2}{dt^2}(r\sin\theta) - r\sin\theta\left(\frac{d\phi}{dt}\right)^2 \text{ and } \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right),$$

the former in the direction NP and the latter perpendicular to the plane CPA.

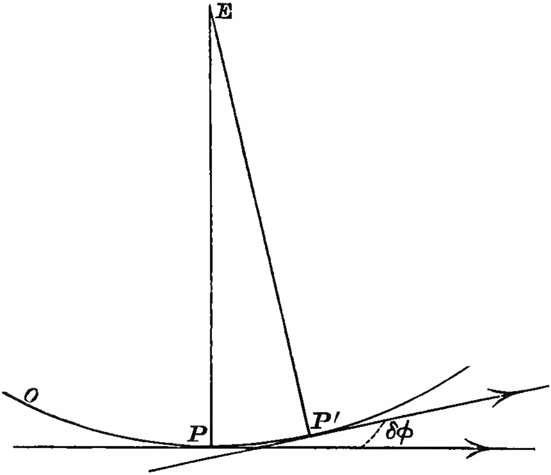
Also the acceleration parallel to Oz

$$= \frac{d^2 . ON}{dt^2} = \frac{d^2}{dt^2}(r\cos\theta).$$

If ρ, τ, σ be the component accelerations in the directions OP, PT perpendicular to OP in the plane CPA, and perpendicular to that plane, it follows, by resolving the above accelerations, that

$$\begin{aligned} \rho &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta, \\ \tau &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta, \\ \sigma &= r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta. \end{aligned}$$

17. *Tangential and normal accelerations of a point moving in a plane curve.*



If v be the velocity at P of a point moving in a curve, and if s be the arc OP of the curve measured from some fixed point O,

$$v = \frac{ds}{dt}.$$

The tangential acceleration at P

$$= \text{Lt of } \frac{(v + \delta v) \cos \delta\phi - v}{\delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

and the normal acceleration

$$= \text{Lt of } \frac{(v + \delta v) \sin \delta\phi}{\delta t} = \frac{v d\phi}{dt} = \frac{v d\phi}{ds} \cdot \frac{ds}{dt} = \frac{v^2}{\rho},$$

if ρ be the radius of curvature at P.

If the motion of the point be uniform motion in a circle, v and ρ are constant, and, if ω be the angular velocity, $v = \omega\rho$. The tangential acceleration is then zero, and the normal acceleration, measured inwards, is equal to $\omega^2\rho$, as in Art. (11).

18. *Accelerations of a point moving in a tortuous curve in directions of the tangent, the principal normal, and the binormal.*

If x, y, z be the coordinates of the point referred to fixed rectangular axes, we have,

$$\frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt},$$

and therefore
$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2} \frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2x}{ds^2},$$

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2y}{ds^2},$$

$$\frac{d^2z}{dt^2} = \frac{d^2s}{dt^2} \frac{dz}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2z}{ds^2}.$$

Now $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction-cosines of the tangent, and if ρ be the radius of absolute curvature,

$$\rho\frac{d^2x}{ds^2}, \quad \rho\frac{d^2y}{ds^2}, \quad \rho\frac{d^2z}{ds^2},$$

are the direction-cosines of the principal normal.

The above equations therefore prove that the resultant acceleration of the point is compounded of the acceleration $\frac{d^2s}{dt^2}$ in direction of the tangent and of the acceleration $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$ in direction of the principal normal.

It follows at once that there is no acceleration in direction of the binormal.

Or, the direction-cosines of the binormal being proportional to

$$\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}, \quad \frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx^2}{ds}\frac{d^2z}{ds^2}, \quad \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2},$$

if we multiply the above equations by these three quantities respectively, and add them together, the right-hand member vanishes identically.

19. These results can also be obtained by the considerations that the osculating plane is the plane containing two consecutive tangents, and that the consecutive osculating plane is obtained by an infinitesimal twist round the tangent.

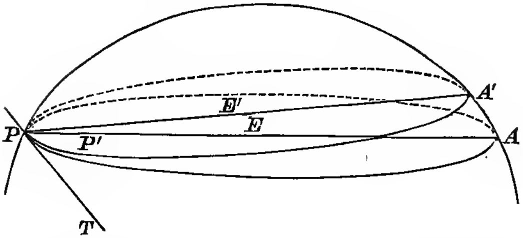
The circles *PA*, *PA′*, in the accompanying figure, are consecutive circles of curvature, the angle between their planes, $\delta\eta$, being the angle of torsion, and the circles being, in general, small circles on the sphere of curvature. The circles may be in certain cases coincident, or either of them may be a great circle.

If P' be a consecutive point on the circle *PA′*, $d\epsilon$ be the angle of contingence, that is, the angle between the tangents at P and P', v the velocity at P, and $v + \delta v$ at P',

the changes of velocity in directions of the tangent *PT*, the principal normal *PE*, and the binormal are

$$(v+\delta v)\cos\delta\epsilon - v,\ \ (v+\delta v)\sin\delta\epsilon\cos\delta\eta,$$

and $$(v+\delta v)\sin\delta\epsilon\,.\,\sin\delta\eta;$$



dividing by δt, we obtain in the limit, the expressions

$$\frac{dv}{dt} \text{ and } \frac{v^2}{\rho}$$

from the first two, and the ratio of the last expression to δt vanishes in the limit.

The Principles of Relative velocities and Relative accelerations.

20. If P be a moving point, A, B, C,...other moving points and O a fixed point, the velocity of P in any direction is the sum of the velocities, in the same direction, of P relative to A, of A relative to B, of B relative to C, and so on, and of the last moving point relative to O.

For if u be the velocity of P in the direction considered, $u_1, u_2, \ldots u_n$ of the moving points in the same direction,

$$u = (u-u_1) + (u_1-u_2) + \ldots\ldots + (u_{n-1}-u_n) + u_n,$$

which establishes the statement.

The same principle is equally true of accelerations,

for $$\frac{du}{dt} = \frac{d}{dt}(u-u_1) + \frac{d}{dt}(u_1-u_2) + \ldots\ldots + \frac{du_n}{dt},$$

that is the acceleration of P is the sum of the relative accelerations.

21. By aid of the foregoing principle, the expressions of Art. (13) are at once obtained.

For $u =$ velocity of P relative to $N +$ velocity of N

$$= -y\dot{\theta} + \dot{x},$$

and so $$v = \dot{y} + x\dot{\theta}.$$

Again, the acceleration of N in the direction Ox

$$= \ddot{x} - x\dot{\theta}^2,$$

and that of P relative to N in the same direction

$$= -(y\ddot{\theta} + 2\dot{y}\dot{\theta});$$

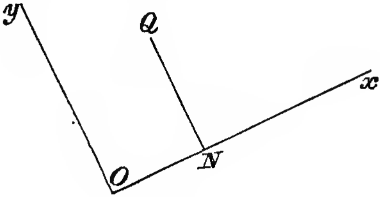
therefore the acceleration of P in the direction Ox

$$= \ddot{x} - x\dot{\theta}^2 - y\ddot{\theta} - 2\dot{y}\dot{\theta};$$

and similarly the acceleration of P in the direction Oy

$$= \ddot{y} - y\dot{\theta}^2 + x\ddot{\theta} + 2\dot{x}\dot{\theta}.$$

22. We can also obtain the expressions for the accelerations in the following manner.



The velocities of a moving point P, at any time t, parallel to two moving directions Ox, Oy can be represented by $$u = ON,\ v = QN,$$

and the accelerations parallel to x and y will be the rates of change of u and v in those directions and will therefore be

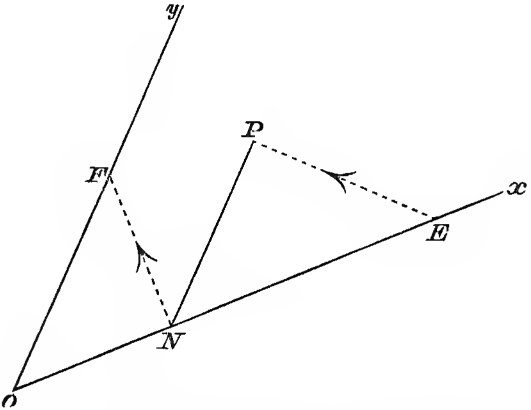
$$\dot{u} - v\dot{\theta}, \text{ and } \dot{v} + u\dot{\theta}.$$

Observing that

$$u = \dot{x} - y\dot{\theta}, \text{ and } v = \dot{y} + x\dot{\theta},$$

we obtain the expressions given at the end of the last article.

23. *Component velocities and component accelerations of a moving point referred to two axes turning round the origin in any given manner.*



Let θ and ϕ be the inclinations, at any instant, of Ox and Oy to some fixed line in the plane, and represent by ω the angle xOy or $\phi - \theta$.

The velocity of P relative to N is $y\dot{\phi}$ perpendicular to PN, and this, by the triangle of velocities EPN, decomposes into
$$-y\dot{\phi}\operatorname{cosec}\omega, \text{ and } y\dot{\phi}\cot\omega,$$
parallel to x and y.

The velocity of N in direction Ox is $\dot{x}$, and perpendicular to Ox is $x\dot{\theta}$.

The latter, by the triangle of velocities ONF, decomposes into
$$-x\dot{\theta}\cot\omega, \text{ and } x\dot{\theta}\operatorname{cosec}\omega,$$
parallel to x and y.

Hence, the velocity of P being compounded of its velocity with regard to N, and of the velocity of N, the components parallel to x and y are respectively
$$\dot{x} - x\dot{\theta}\cot\omega - y\dot{\phi}\operatorname{cosec}\omega,$$
$$y + x\dot{\theta}\operatorname{cosec}\omega + y\dot{\phi}\cot\omega.$$

In exactly the same manner the component accelerations are at once seen to be

$$\ddot{x} - x\dot{\theta}^2 - (x\ddot{\theta} + 2\dot{x}\dot{\theta}) \cot \omega - (y\ddot{\phi} + 2\dot{y}\dot{\phi}) \operatorname{cosec} \omega,$$
$$\ddot{y} - y\dot{\theta}^2 + (x\ddot{\theta} + 2\dot{x}\dot{\theta}) \operatorname{cosec} \omega + (y\ddot{\phi} + 2\dot{y}\dot{\phi}) \cot \omega.$$

24. The foregoing expressions can be obtained in a different manner.

Thus if u and v be the component velocities,

$$u \cos \theta + v \cos \phi = \frac{d}{dt} (x \cos \theta + y \cos \phi),$$

$$u \sin \theta + v \sin \phi = \frac{d}{dt} (x \sin \theta + y \sin \phi),$$

and if f, f' be the component accelerations,

$$f \cos \theta + f' \cos \phi = \frac{d^2}{dt^2} (x \cos \theta + y \cos \phi),$$

$$f \sin \theta + f' \sin \phi = \frac{d^2}{dt^2} (x \sin \theta + y \sin \phi).$$

The solution of these equations will give the expressions for u, v, f and f'.

25. *Particular illustrations of the use of the principle of relative velocities and relative accelerations.*

(1) *A point P describes an equiangular spiral with uniform angular velocity round O, and a point Q describes an equal spiral with the same angular velocity round P; it is required to find the path of Q.*

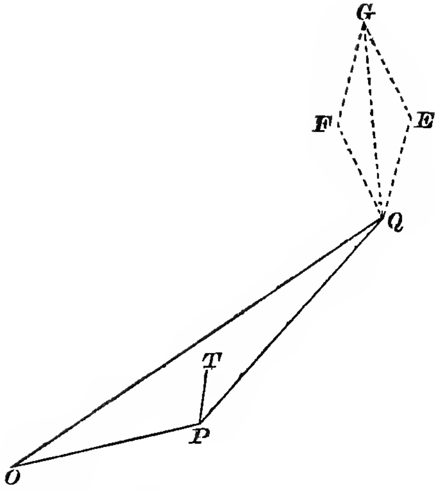
Since $r = a\epsilon^{\theta \cot \alpha}$, we have

$$\dot{s} = \sec \alpha \,.\, \dot{r} = r\dot{\theta} \operatorname{cosec} \alpha,$$

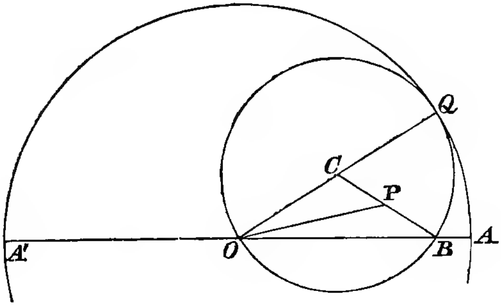
so that, if μOP be the velocity of P, in the direction PT, μPQ is the velocity of Q relative to P in the direction QF, the angles OPT, PQF being equal and constant.

Hence μOQ is the actual velocity of Q.

Further if QE, QF represent the component velocities in direction and magnitude, the resulting velocity of Q is represented by QG the diagonal of the parallelogram, and the angle $OQG = PQG - PQO = PQF + FQG - EGQ = PQF$; hence it follows that the path of Q is an equal spiral.



(2) *A circle rolls uniformly inside a circle of double its radius; it is required to find the acceleration of any carried point.*



If P be the carried point on the radius CB, and if OB produced meet the circle in A, it follows, since the angle $QCB = 2\,COB$, that the arcs QA, QB are equal.

Therefore A is a fixed point and B moves along the line AOA'.

Since OC, CB make equal angles with OA it follows that the angular velocities of the lines OC, CP are equal.

Hence, if ω be this angular velocity, the acceleration of C is in the direction CO and $= \omega^2 . CO$, and the acceleration of P relative to C is in the direction PC and $= \omega^2 . PC$;

therefore, by the triangle of accelerations, the resulting acceleration of P is in the direction PO and $= \omega^2 . PO$.

(3) *A circle rolls on a straight line; it is required to find the acceleration of the point of the circle in contact with the line.*

If θ be the angle through which the circle has rolled from any assigned position, and a the radius of the circle, $a\theta$ is the linear space traversed by the centre C and therefore $a\ddot{\theta}$ is the acceleration of the centre.

The accelerations of the point of contact P, relative to the centre are $a\dot{\theta}^2$ in direction PC, and $a\ddot{\theta}$ parallel to the line and in the direction opposite to that of the motion.

Compounding these with the acceleration of C, it results that the acceleration of P is in direction PC and is equal to $a\dot{\theta}^2$, or to $a\omega^2$, if ω be the angular velocity.

This result may otherwise be obtained as follows:

Let x and y be the coordinates of the point initially in contact; then

$$x = a\theta - a \sin\theta, \text{ and } y = a - a\cos\theta,$$

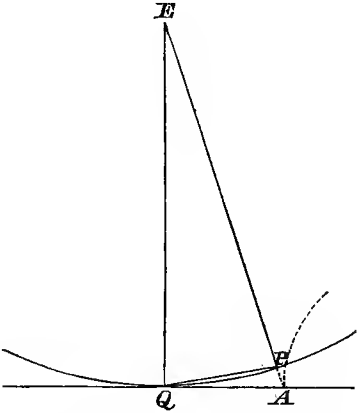
from which $\ddot{x} = a\ddot{\theta} - a(\cos\theta . \ddot{\theta} - \sin\theta . \dot{\theta}^2)$,

$$\ddot{y} = a(\sin\theta . \ddot{\theta} + \cos\theta . \dot{\theta}^2),$$

and, putting $\theta = 0$, $\ddot{x} = 0$, and $\ddot{y} = a\dot{\theta}^2$,

we hence infer that if ρ be the radius of curvature, at the point of contact, of any curve rolling on a straight line, the acceleration of the point of the curve in contact with the line is $\omega^2\rho$ in the direction perpendicular to the line.

Or we can give a proof directly from the definition of acceleration.



Let the curve PQ roll on a straight line, the point P rising from the point A.

When at A, P has no velocity, and when Q is the point, of contact, and is therefore the instantaneous centre, the velocity of P is $\omega . PQ$ perpendicular to PQ.

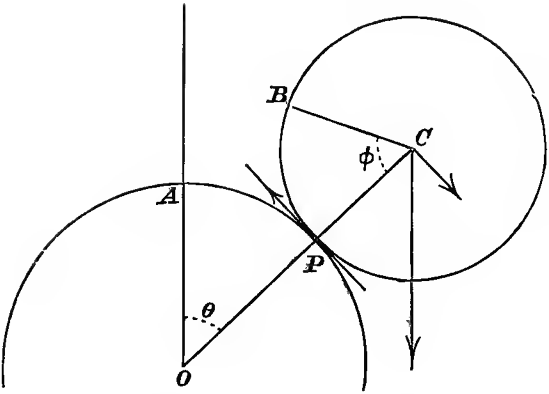
Taking PQ as an infinitesimal arc, and E as the centre of curvature, the velocities of P parallel and perpendicular to the line are $\omega . PQ \sin \delta\theta$ and $\omega . PQ \cos \delta\theta$, if $PEQ = \delta\theta$.

Hence if δt be the time the accelerations are the limits of $\omega\, PQ \dfrac{\sin \delta\theta}{\delta t}$ and $\omega\, PQ \dfrac{\cos \delta\theta}{\delta t}$, and, as $PQ = \rho\delta\theta$, the first of these ultimately vanishes, and the second $= \omega^2\rho$.

(4) *A circle of radius b rolls on a circle of radius a; it is required to determine the acceleration of the point of contact.*

Taking the arc BP equal to the arc AP, and ω as the angular velocity,

$$\omega = (\dot{\phi} + \dot{\theta}) = \frac{a+b}{b}\dot{\theta},$$



and the accelerations of P relative to C, in the direction PC and perpendicular to it, are

$$b\omega^2 \text{ and } b\dot{\omega}.$$

The accelerations of C, in the direction CO and perpendicular to it, are

$$(a+b)\,\dot{\theta}^2 \text{ and } (a+b)\,\ddot{\theta},$$

or
$$\frac{b^2\omega^2}{a+b} \text{ and } b\dot{\omega}.$$

Compounding these with the relative accelerations of P, it results that the acceleration of P is in the direction PC and is equal to

$$\frac{ab}{a+b}\,\omega^2.$$

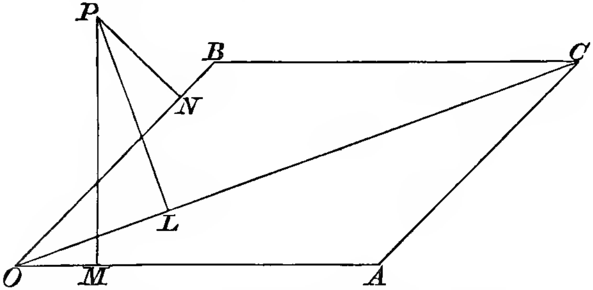
Replacing a and b by radii of curvature this expression gives the acceleration of the point of contact of any curve rolling on a fixed curve.

26. We have considered in Art. (7) the angular velocity of a line moving in a plane; we shall now find it necessary to consider the angular velocity of a rigid system of lines and points in space.

The angular velocity of a rigid system about an axis is the rate of increase of the inclination of a plane fixed in the system, passing through the axis, to a plane passing through the axis, fixed in space.

Parallelogram of angular velocities.

Imagine that a rigid system has two coexistent angular velocities ω, ω' about two axes OA, OB. Construct a parallelogram $AOBC$ such that OA and OB are in the ratio of the angular velocities, and take a quantity Ω such that



$$OA : OB : OC :: \omega : \omega' : \Omega.$$

If P be any point in the plane AOB, the velocity of P, due to the two angular velocities, is equal to

$$\omega\, PM + \omega' PN,$$

perpendicular to the plane.

Now, $OACB$ being a parallelogram, we know that

$$PL \,.\, OC = PM \,.\, OA + PN \,.\, OB;$$

$$\therefore \qquad \Omega \,.\, PL = \omega \,.\, PM + \omega' PN,$$

and therefore the velocity of $P = \Omega \,.\, PL$, which is the velocity due to an angular velocity Ω about OC.

The line OC therefore represents the resultant angular velocity.

Hence it follows that angular velocities are subject to the parallelogrammic law, and can be compounded and decomposed in the same way as linear velocities.

In other words, an angular velocity is a vector.

27. If a rigid system be in motion about a fixed point, there is always one line in the system which has no motion and about which the system is turning.

It is clear that the motion of the system is completely determined by the motions of any two given lines OP, OQ of the system. Now, at any instant, OP must be moving in some plane and therefore must be turning round some straight line in the plane through OP perpendicular to the plane of motion of OP.

Similarly OQ must be turning round some line in the plane through OQ perpendicular to the plane of motion of OQ.

If then OC be the line of intersection of the two planes through OP and OQ, perpendicular respectively to their planes of motion, the motion of the system is completely represented by a state of rotation about OC.

Any state of motion of a rigid system about a fixed point can therefore be represented by a single angular velocity, or by three coexistent angular velocities about three lines through the fixed point.

28. *Velocities and accelerations of a point referred to three moving axes at right angles to each other.*

Let θ_1, θ_2, θ_3 represent, at any instant, the angular velocities of the system of axes about the axes themselves, or rather, about the lines fixed in space with which the axes are, at the instant, coincident.

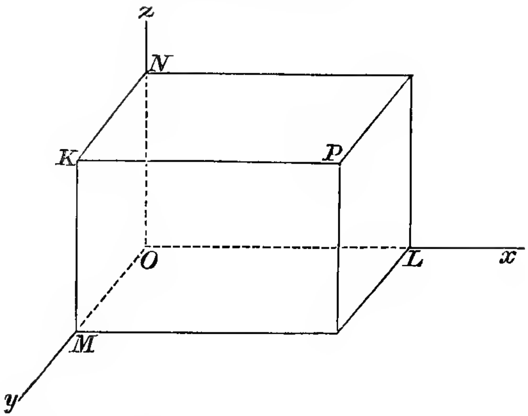
If u, v, w be the component velocities,

$u =$ velocity of P relative to $K +$ that of K relative to N $+$ that of N relative to O

$$= \dot{x} - y\theta_3 + z\theta_2,$$

and similarly

$$v = \dot{y} - z\theta_1 + x\theta_3,$$
$$w = \dot{z} - x\theta_2 + y\theta_1.$$



For the accelerations,

let $$OL = u, \ \ OM = v, \ \ ON = w$$

represent the component velocities;

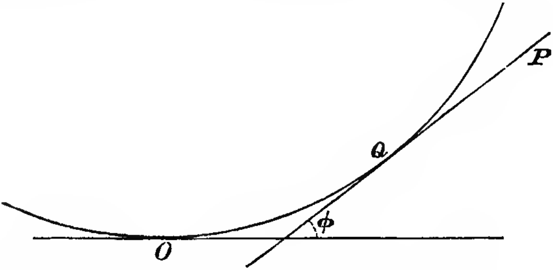
then the accelerations parallel to the axes are, on this scale, the velocities of P, and are therefore

$$\dot{u} - v\theta_3 + w\theta_2$$
$$\dot{v} - w\theta_1 + u\theta_3$$
$$\dot{w} - u\theta_2 + v\theta.$$

Or, the acceleration of P relative to N, in the direction Ox, being $\dot{u} - v\theta_3$, and the acceleration of N in the same direction being $w\theta_2$, the acceleration of P parallel to Ox is the sum of these two, and the accelerations parallel to Oy and Oz are obtained in the same manner.

29. If the position of a moving point P be defined by the length r of the tangent PQ to a given curve, and the deflection ϕ of the tangent,

the acceleration parallel to PQ relative to $Q = \ddot{r} - r\dot{\phi}^2$,
and that of Q in the same direction $= \ddot{s}$, if the arc $OQ = s$,
$\therefore$ acceleration of P in the direction $QP = \ddot{r} - r\dot{\phi}^2 + \ddot{s}$

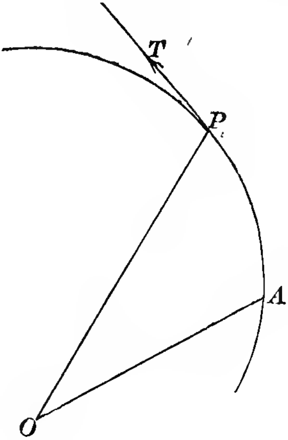


and, similarly, acceleration perpendicular to PQ

$$= r\ddot{\phi} + 2\dot{r}\dot{\phi} + \frac{1}{\rho}\dot{s}^2,$$

if ρ be the radius of curvature at Q.

30. *A point moves on a given curve, while the curve turns round a fixed point in its plane: it is required to find expressions for the accelerations of the point.*



If $OP = r$, the accelerations of the point P of the curve are $\omega^2 r$ and $r\dot{\omega}$ in the direction PO and perpendicular to PO.

If the moving point be passing over the point P its accelerations are those of P compounded with its accelerations relative to P.

These relative accelerations are due to the angular motion of PT and to the motion on the curve, and, in the respective directions of the tangent PT and the normal at P, are, if s be the arc AP measured from a given point A of the curve,

$$\ddot{s} \text{ and } 2\omega\dot{s} + \frac{\dot{s}^2}{\rho},$$

in accordance with the observation of Art. (10).

It is an instructive exercise to obtain these expressions by taking a consecutive position of the moving point on the curve, when twisted through a small angle $\omega\delta t$, and actually resolving its velocities, relative to P, in the directions PT and perpendicular to it.

EXAMPLES.

1. A railway train is moving at the rate of 30 miles an hour, when it is struck by a stone moving horizontally and at right angles to the train with the velocity of 33 feet per second. Find the magnitude and direction of the velocity with which the stone appears to meet the train.

2. Assuming that the earth describes a circle uniformly about the sun in a year, that the distance of their centres is 240 radii of the sun, and that the radius of the sun is 100 times that of the earth, find the measure of the velocity of the vertex of the earth's shadow, taking the sun's radius as the unit of length and a year as the unit of time.

3. If one point move uniformly in a circle, and another move with equal velocity in a tangent to the circle, what are their relative paths?

4. The radius of the earth being 4000 miles, the latitude, λ, of a place at which a train travelling westward at the rate of 1 mile per minute is at rest in space is given by

$$\cos \lambda = \frac{9}{50\pi}.$$

5. A circle revolves with uniform velocity about its centre. The centre moves with varying velocity along a straight line. Find the velocity parallel to this line at any instant of a point on the circumference, and deduce the acceleration of the centre necessary for this point to be always moving at right angles to the line.

6. A point moves in a curve in such a way that its direction of motion changes at a rate varying as the velocity directly and the whole space described inversely. Prove that the curvature varies inversely as the arc.

7. A wheel revolves uniformly about its centre C, which is fixed, and a particle A moves uniformly in a straight line through the centre; describe the path of a point B in the wheel relative to A, (1) when CA is in the plane of the wheel, (2) when CA is perpendicular to that plane.

8. If the resolved parts of the velocity of a moving particle perpendicular to its distances from two fixed points are constant, and equal to one another, its velocity varies as the square root of the product of its distances from these points.

9. A particle A moves in a straight line, a second particle B always moves towards A and keeps at a constant distance from it. Find the path of B, and shew that its velocity is a mean proportional between the velocity of its projection on the path of A and the velocity of A.

10. If a point be situated at the intersection of the perpendiculars, drawn from the angular points of a triangle to the sides respectively opposite to them, and have three com-

ponent velocities, represented, in magnitude and direction, by its distances from the angular points of the triangle, prove that its resultant velocity will tend to the centre of the circle circumscribing the triangle, and will be represented by twice the distance of the point from the centre.

11. The tangent at a point P of a parabola meets the tangent at the vertex in Y and the axis in T. If Y move with uniform velocity, shew that T moves with uniform acceleration: if T move with uniform velocity, the velocity of Y varies inversely as AY.

12. If the velocity of a point be resolved into any number of components in a plane, its angular velocity about any fixed point in the plane is the sum of the angular velocities due to the several components.

13. A point moves in a plane curve and sounds as it moves. At a fixed point C in the plane the whole sound produced is heard simultaneously. Shew (i) that if the point moves uniformly, the curve is an equiangular spiral—(ii) if the velocity of the point vary inversely as the distance of C from its line of motion, the curve is a reciprocal spiral.

14. A point moves in the arc of a cycloid so that the tangent turns uniformly; prove that the acceleration of the point is constant.

15. If the axes Ox, Oy revolve with constant angular velocity ω, and the component velocities of the point (xy) parallel to the axes are

$$\frac{a^2-b^2}{a^2+b^2}\omega y,\ \frac{a^2-b^2}{a^2+b^2}\omega x,$$

prove that the point describes relatively to the axes an ellipse in the periodic time

$$\frac{\pi}{\omega}\cdot\frac{a^2+b^2}{ab}.$$

Prove that the locus of the points about which the angular velocity of a point moving in any manner is, at the same instant, the same, is a circle.

16. If the acceleration of a falling body be the unit of acceleration and a velocity of 60 miles an hour the unit of velocity, find the units of length and time.

17. If the angular velocity of a particle about a given point in its plane of motion be constant, prove that the transversal component of its acceleration is proportional to the radial component of its velocity.

18. If the acceleration of a falling body be the unit of acceleration, and if a velocity of a yard per minute be the unit of velocity, find the units of space and time.

19. If a lamina move in its plane so that two fixed points in it describe straight lines with accelerations f, f', shew that the acceleration of the centre of instantaneous rotation is

$$\frac{\sqrt{f^2 + f'^2 - 2ff' \cos\theta}}{\sin\theta},$$

θ being the angle between the lines.

20. A lamina moves in its own plane so that two points fixed in the lamina describe straight lines with equal accelerations; prove that the acceleration of the centre of instantaneous rotation is constant in direction.

21. A point P moves with uniform velocity in a circle; Q is a point in the same radius at double the distance from the centre, PR is a tangent at P equal to the arc described by P from the beginning of the motion; shew that the acceleration of the point R is represented in magnitude and direction by RQ.

22. In two different systems of units an acceleration is represented by the same number, while the velocity is represented by numbers in the ratio 1 : 3. Compare the units of time and space.

23. If the time is a quadratic function of the space described, prove that the tangential acceleration is proportional to the cube of the velocity.

24. A point moves in an ellipse so that the velocity varies as the square of the diameter parallel to the direction of motion: prove that the resultant acceleration at any instant will be in direction of the line joining the point with the middle point of the perpendicular from the centre on the tangent at the point.

25. A point moves in a plane in such a manner that its tangential and normal accelerations are always equal, and its velocity varies as $\epsilon^{\tan^{-1}\frac{s}{c}}$, s being the length of the arc of the curve measured from a fixed point; find the path.

26. If a curve roll in contact with a straight line with uniform velocity, shew that the acceleration of the point in contact with the straight line varies inversely as ρ, but if with uniform angular velocity directly as ρ; ρ being the radius of curvature of the curve at the point in contact.

27. A curve rolls along a straight line, the point of contact moving uniformly along the line. Shew that the acceleration of the centre of curvature of the rolling curve at the point of contact is proportional to $\frac{d^2\rho}{ds^2}$ in the curve at the point.

28. If the motion be referred to two axes one of which is fixed, and the other revolves about the origin in such a way that the line joining the origin to the particle is equally inclined at an angle $\frac{1}{2}\theta$ to the axes, shew that the component acceleration parallel to the fixed axis (ξ) is

$$\ddot{\xi} - (2\dot{\xi}\dot{\theta} + \xi\ddot{\theta}) \operatorname{cosec} \theta.$$

What is the other component?

29. If the radial and transversal accelerations of a particle be each proportional to the velocity in the direction of the other, the path of the particle is given by the equation of the form

$$\left(1 - \frac{a^2}{r^2}\right)^{-2} \left(\frac{dr}{d\theta}\right)^2 = Ar^2 + B \log r + \frac{C}{r^2}.$$

30. If the perpendiculars from a point P on axes Ox, Oy of which Ox is fixed, and Oy revolves uniformly, are ξ, η respectively, prove that the accelerations of P parallel to the axes at the instant when they are perpendicular are $\ddot{\eta}-2\dot{\xi}\omega+\eta\omega^2$, and $\ddot{\xi}$; the angular velocity of Oy being ω.

31. The centre C of an elliptic wire is moved with uniform velocity along a fixed line CY in its own plane whilst the wire is in contact at P with a fixed line PY perpendicular to CY; shew that the acceleration perpendicular to PY of the point of the curve in contact at $P \propto x^{-2}y^{-2}p^{-5}$, where $CY=p$, and x, y are the co-ordinates of P referred to the axes.

32. A point moves in a plane with an angular velocity ω, and the plane is turning round the radius vector with an angular velocity ω'; prove that the accelerations in the plane are $\ddot{r}-\omega^2r$, and $r\dot{\omega}+2\dot{r}\omega$, and that the acceleration perpendicular to the plane is $r\omega\omega'$.

33. A point P moves on a straight line OP which is made to describe uniformly a right circular cone about an axis OA, while OA sweeps out uniformly a right circular cylinder; find an expression for the acceleration of the point P in the direction OP.

34. A point P moves so that its velocity is compounded of two constant velocities, one of which is in a fixed direction and the other is perpendicular to the line joining P to a fixed point. Find the orbit described by P.

35. A plane is moving about an axis perpendicular to it, and a point is moving in a given curve traced on the plane; in any position ω is the angular velocity of the plane, v the velocity of the particle relative to the plane, r its distance from the axis, p the perpendicular on the tangent, s the arc described along the plane, prove that the acceleration along the tangent to the curve is

$$v\left(\frac{dv}{ds}+p\frac{d\omega}{ds}\right)-\omega^2r\frac{dr}{ds}.$$

36. The velocity of a point moving in a plane is the resultant of two velocities v and v' along two radii vectores r and r' measured from two fixed points at a distance a apart. Prove that the corresponding accelerations are

$$\frac{dv}{dt}+\frac{vv'}{2r^2r'}(r^2-r'^2+a^2),$$

and
$$\frac{dv'}{dt}+\frac{vv'}{2rr'^2}(r'^2-r^2+a^2).$$

37. A circle of radius a rolls on a second fixed circle of the same radius and a point P on it traces out a cardioid. If O and C be the centres respectively of the moving and fixed circles, prove that the accelerations of P along and perpendicular to PO are

$$2a(2\dot{\theta}^2-\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta),$$

$$4a\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}\ddot{\theta}-\dot{\theta}^2\cos\frac{\theta}{2}\right),$$

θ being the angle OC makes with a fixed line.

38. Two circles are taken, and the motion of a point is given by the component velocities (u, u') in the directions of two tangents drawn one to each circle. Shew that the component accelerations in the same directions are respectively,

$$\left.\begin{aligned}&\frac{du}{dt}+uu'\left(\frac{1}{l'}-\frac{\cos\phi}{l}\right)\\ \text{and}\quad&\frac{du'}{dt}+uu'\left(\frac{1}{l}-\frac{\cos\phi}{l'}\right)\end{aligned}\right\},$$

where (l, l') are the lengths of the tangents, and (ϕ) their mutual inclination.

CHAPTER IV.

31. THE preceding discussions belong to the domain of pure reason; we have now to introduce the facts of nature, and to employ the results we have obtained in the solution of actual cases of motion.

For this purpose the laws enunciated by Newton are sufficient, and, once enunciated, the solution of any problem concerning the motion of a body or a system reduces itself to the integration of differential equations of the second order, and the interpretation of the solutions.

The introduction of the principles of momentum and of energy will in many cases enable us to determine the motion of a body or a system in a simple manner and without the intervention of differential equations of the second order.

32. A particle of matter is supposed to be a very small body, but possessing a sensible mass and capable of being acted upon by forces, and, for theoretical purposes, two particles are supposed to differ from each other only in the case of their having different masses.

Force is any cause which tends to change the state of rest or motion of a particle or a body. The weight of a body for instance is the force of the action of gravity upon it, and is found experimentally to be proportional to the mass or to the quantity of matter in the body.

Experiments cannot be made with particles, such as we have imagined, but experiments made with bodies of various shapes and sizes lead to the enunciation of, and belief in the laws of motion as applied to a particle, or to a body of any kind; and the results of theoretical calculations, tested by experiment, and applied, on a large scale, to the motions of Planetary Bodies, have led to a profound conviction, in the minds of students of mechanical science, of the truth of these laws.

THE LAWS OF MOTION.

33. *First Law of Motion.*

If a particle be not acted upon by any external forces, it is either at rest or it moves uniformly in a straight line.

Second Law of Motion.

Change of motion is proportional to the acting force, and is in the direction in which the force is acting.

Third Law of Motion.

Action and Reaction are equal and opposite.

34. With regard to the first of these laws, it is only necessary to remark that it is confirmed by the perpetual experiences of all ordinary phenomena.

Any change of motion of a body is seen to be due to the action of some force, and the more we can eliminate the action of external force the more nearly we find that the motion of a body approaches to that of uniform motion in a straight line.

35. The Second Law contains really two distinct statements.

The first is the enunciation of the principle of the *physical independence of forces*, namely, that each force produces its full effect in its own direction.

To illustrate this consider the case of a ball receiving, simultaneously, two impulses in different directions. Supposing the velocity due to each individual impulse, when applied alone, to be known, then the second law tells us that the ball is at once imbued with two coexistent and given velocities in given directions, and the motion of the ball, due to the two simultaneous impulses, is immediately determined by the parallelogram of velocities.

The second is the quantitative relation between the magnitude of the acting force and the change of motion produced.

Definitions. If m be the mass of a particle and v its velocity, the product mv is called its momentum or quantity of motion, and the rate of change of momentum is $m\dot{v}$ or mf, if f is the acceleration of the particle.

The assertion of the second law is that the momentum produced in some given time is proportional to the magnitude of the acting force. If a constant force P acting on a body of mass m, produce in the unit of time the velocity f, mf is the momentum acquired and therefore

$$P \propto mf,$$

and, if we choose the units so that the unit of mass is that in which the unit of force produces the unit of acceleration, we obtain

$$P = mf.$$

This equation really contains all the kinetics of a single particle.

In any case whether the force be constant or variable, it is proportional to the momentum which it is capable of producing in a given time, or, which is the same thing, to the rate of change of momentum, and therefore, in general

$$P = mf.$$

Even if the mass which is acted upon be variable, the effect of force upon it is the production of momentum, and the measure of the force is the rate of that production, so that

$$P = \frac{d}{dt}(mv),$$

or, in other words, force is the time-flux of the momentum.

36. *Impulses.* We have spoken of an impulse as an action producing velocity instantaneously in a given mass,

and such an action is proportional to the momentum which is apparently at once produced by it, so that if Q be the measure of an impulse producing a velocity v in a mass m,

$$Q = mv.$$

There is no real difference between a momentum produced gradually and a momentum produced in a very short interval of time.

The application of a powerful time-microscope to the latter case would present the appearance of a force gradually accumulating momentum, and the final result is that which is spoken of as being instantaneously produced.

If P be the measure of a very large and variable force acting for a short time τ, and producing the momentum mv,

then $$mv = \int_0^\tau P dt.$$

If in any given case we could find this short time τ, then the mean value of the measure of the force would be mv/τ.

The expression $\int_0^\tau P dt$, or the time integral of the force, is called the impulse of the force, or, briefly, the impulse. If the time τ be infinitely small, and the force P infinitely large, a finite momentum mv may be instantaneously acquired.

37. In the particular case of the action of gravity, if W be the weight of a body, and g the acceleration of a falling body, it follows that

$$W = mg.$$

From this it appears if we take a pound as the unit of mass, the weight of a pound is g units of force, so that, g being 32·2 when a foot and a second are units, the unit of force is approximately equal to the weight of half an ounce. This is the British absolute unit of force.

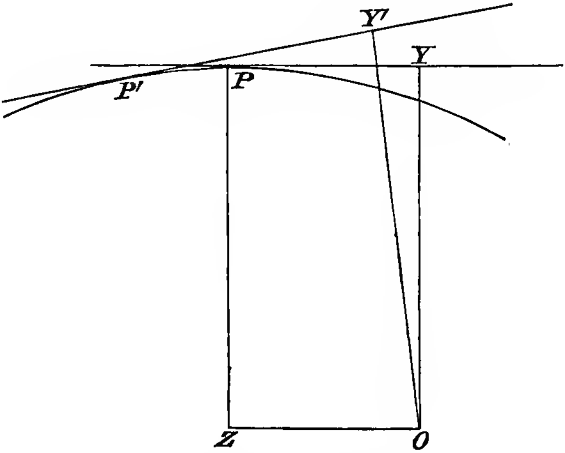
38. *Parallelogram of forces.* If a particle be acted upon by two known forces in given directions, we know from the second law that the particle has coexistent, in these directions, two known accelerations.

The parallelogram of accelerations gives the resultant of these two, and therefore it follows that the resultant of the forces follows the same law, and the parallelogram of forces is at once established.

39. The *angular momentum* of a particle about a fixed axis is the moment of its momentum about the axis; and therefore, if v be the component, perpendicular to the axis, of the velocity of the particle, p the distance between the axis and the line of this velocity, and h the angular momentum,

$$h = mvp.$$

Now consider the motion of a particle in a plane, and let T and N be the acting forces in directions of the tangent and normal to its path.



The time-flux of the angular momentum about the axis perpendicular to the plane through the fixed point O

$$= \dot{h} = m\dot{v}p + mv\dot{p}.$$

The first term of this expression, $m\dot{v}p$, is equal to Tp and represents the moment about the axis of the tangential force.

During an infinitesimal period of time δt, the velocity has changed to $v+\delta v$, and the direction of motion has turned through an angle $\delta\phi$, and therefore the change of momentum in direction of the normal PZ is $m(v+\delta v)\sin\delta\phi$, or $mv\delta\phi$.

Now, if OZ be the perpendicular on the normal

$$v\delta p = v \,.\, OZ \,.\, \delta\phi,$$

and therefore
$$mv\delta p = mv\delta\phi \,.\, OZ.$$

But, N being the normal force $N\delta t$ is the change of momentum in direction of PZ, and therefore

$$N\delta t = mv\delta\phi, \text{ or } N = mv\dot{\phi}$$

and consequently
$$mv\dot{p} = mv\dot{\phi} \,.\, OZ = N \,.\, OZ,$$

which is the moment of the normal force.

Hence we obtain

$$\dot{h} = T \,.\, OY + N \,.\, OZ,$$

that is, the time-flux of the angular momentum is equal to the moment of the acting forces.

In the case of motion in a tortuous curve, or in any manner in three dimensions, the same result is true if the figure represent the projection, on a plane perpendicular to the axis, of the path of the particle.

Other methods may be adopted to obtain this result.

For instance, we can transform to polar coordinates, by observing that $p\delta s = r^2\delta\theta$, each being the double of the area of an infinitesimal triangle, so that

$$h = mp\dot{s} = mr^2\dot{\theta},$$

and
$$\dot{h} = m\frac{d}{dt}(r^2\dot{\theta}).$$

But we know that $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ is the expression for the transversal acceleration, and therefore, if Q be the transversal force in action,

$$Q = m \,.\, \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}),$$

so that $$\dot{h} = Q \cdot r,$$
and Qr is the moment of the acting forces.

If A be the vectorial area, or the area swept over by the radius vector,

$$\dot{A} = \tfrac{1}{2} r^2 \dot{\theta},$$

and therefore $$h = 2m\dot{A},$$

and, $\dot{A}$ being the rate at which area is being swept over per unit of time, h is the product of a mass and an area, and also, being a vector, or directed quantity, it can be represented by a straight line, and is subject to the parallelogrammic law.

40. The third law expresses the fact that if two bodies act on each other in any way, either by contact, or through a connection by means of strings or rods, or by mutual attraction or repulsion, the force which one body exerts on another is exactly the same in amount but opposite in direction to that which the other body exerts on the one.

Some important theorems are at once deducible from the third law, which are of the greatest utility in the discussion of the motion of systems of particles or bodies.

Take the case of a system of bodies, attracting or repelling each other, acting on each other by contact for a finite time, or by mutual impulse. In this case any momentum which is produced or destroyed in any assigned direction is accompanied by the production or destruction of an equal momentum in the contrary direction.

Hence it follows that, if no extraneous forces act on a system of bodies, the total momentum of the system in any assigned direction remains constant.

This is the principle of the conservation of linear momentum.

Again the moments of two equal and opposite momenta about any straight line fixed in space being equal and opposite in direction, *it follows that the angular momentum about any given axis, which is defined to be the sum of the moments of momenta of all the particles of the system, remains constant provided that no external forces act on the system.*

This is the principle of the conservation of angular momentum.

41. One immediate consequence of the preceding results is that if no extraneous forces act on a system, its centre of inertia is either at rest or moves uniformly in a straight line.

For if ξ be the distance from any fixed plane of the centre of inertia

$$\Sigma (m) . \xi = \Sigma (mx),$$

x being the distance from the plane of a particle m.

This gives

$$\Sigma (m) \dot{\xi} = \Sigma (m\dot{x}),$$

or $\Sigma (m) . \dot{\xi}$ is equal to the total momentum perpendicular to the plane, and as this is constant, $\dot{\xi}$ is constant; and the same thing is true of any other direction.

42. Again, as any acting force produces momentum in its own direction, it follows that the sum of the forces acting on a system in any assigned direction is equal to the rate of the change, that is to the time-flux, of the total momentum of the system in that direction; and that, for impulsive actions, the sum of the impulses in any direction is equal to the immediate change, in that direction, of the momentum of the system.

Further, since the aggregate of the forces which are at any instant acting on the particles of a system have for their resultant the system of extraneous acting forces, the moments of these two systems about any fixed axis are equal.

But the former are producing changes of angular momenta, and their moments about the fixed axes are equal to the rates of change of the angular momenta of the particles.

Therefore the sum of the moments, about any fixed axis, of the extraneous acting forces is equal to the rate of change, that is, to the time-flux, of the angular momentum of the system about that axis.

For impulsive actions, the sum of the moments of the extraneous impulses about any fixed axis is equal to the immediate change in the angular momentum of the system about that axis.

43. We now see that the principle of the conservation of angular momentum, as stated in Art. (40), should be, more generally, given as follows.

If the extraneous forces, acting on a system, have no moment about a given axis, the angular momentum of the system about that axis remains unchanged.

44. These *principles of motion*, which are derived immediately from Newton's laws, constitute the whole of the Kinetics of a system.

If Q be the linear momentum of a system in any direction, and P the sum of the acting forces in that direction, the connecting equation is

$$\dot{Q} = P.$$

If H be the angular momentum about any assigned axis, fixed in space, and L the sum of the moments of the acting forces about that axis, the connecting equation is

$$\dot{H} = L.$$

For impulsive actions, the corresponding equations will be, if Q', H' be the new values of Q and H,

$$Q' - Q = K, \text{ and } H' - H = G,$$

K and G being the sums, in the directions considered, of the impulses and impulsive couples.

It will be seen that linear momentum and angular momentum are quantities of the nature of vectors; that is, they can be represented by straight lines, and are subject to the parallelogrammic law.

45. A rigid body is considered to be an aggregation of particles, or molecules, bound together by the forces of internal mutual attractions which are in all cases equal and opposite.

It follows therefore from the preceding articles that the linear and angular momenta imparted to a rigid body by any extraneous forces are independent of the shape, size, or nature of the body, but depend only on the acting forces; and hence that the motion of the centre of gravity of a body is the same as if it were a single particle into which is concentrated the mass of the body, and the rotation of the body about the centre of gravity is independent of the motion of that point, and depends on the moments, about axes through it, of the acting forces.

46. We can now state, in general terms, the principle embodying the preceding discussions.

The forces affecting the particles of a body or system of any kind, are the extraneous acting forces, and the internal forces, due to mutual pressures, or to mutual repulsions or attractions, and these systems together are the exact equivalents of the system of time-fluxes of momenta.

But the system of internal forces, which is made up of sets of equal and opposite forces, has no resultant, and therefore it follows that

The system of time-fluxes of momenta is the exact equivalent of the system of acting forces.

47. *Energy* is capacity for doing work, and a system may possess energy of motion, or energy of position, or both, the former being due to the relative motions of the bodies which constitute the system, and the latter to their relative positions.

The energy of motion is called *Kinetic energy* and is measured by the expression $\frac{1}{2}\Sigma(mv^2)$, v being the velocity of a body, m, of the system.

As applied to a single particle in a field of force, if P be the acting force, $P = m\dot{v} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds} = \frac{1}{2}\frac{d}{ds}(mv^2)$,

so that force is measured by the space-flux of the kinetic energy.

The energy of position, or *the Potential energy*, of a system is the work which it is capable of doing in virtue of its configuration, that is, the relative position of the bodies of the system, or it is the work which has been expended upon it in moving it from a certain defined configuration to its present configuration, it being understood the work done by a force is the product of the force by the space through which it is exerted.

Suppose for instance that two equal spherical balls, each of mass m and radius a, attract each other with a constant force P, and that they are placed at rest with their centres at a distance $2c$.

If the zero configuration be when they are in contact, the work $P(2c-2a)$ must have been done to separate them. This is the initial potential energy of the system of the two balls, their kinetic energy being initially zero.

The system being left to itself, the balls will approach each other, and when their centres are at a distance $2x$, the kinetic energy of each is $\frac{1}{2}m\,.\,2\frac{P}{m}(c-x)$, and therefore the kinetic energy of the system is $2P(c-x)$. But in this configuration the potential energy of the system is $P(2x-2a)$. The sum of the kinetic and potential energies is therefore $P(2c-2a)$, and is constant during the motion.

For another instance take the case of two balls connected by an elastic string and pulled apart; potential energy is thus stored up, and if the balls be let go, kinetic energy is acquired which is the exact equivalent of the loss of potential energy due to the contraction of the string.

48. The preceding is a very simple case of a great general principle, which we now proceed to enunciate.

The Principle of energy. In any conservative system the sum of the kinetic and potential energies is a constant quantity.

As applied to a mechanical system, the meaning is that, in a conservative system, there is no loss of energy by conversion of kinetic energy into heat, or by internal friction producing loss of kinetic energy without gain of potential energy.

There is no doubt that in any system the principle is universally true aud that, in all cases, any loss of ordinary mechanical energy is accounted for by its conversion into heat or some other form of energy.

In other words we may say that the gain of kinetic energy of a system is equal to the work done by the forces of the system, which is in effect the loss of potential energy.

49. For a single particle, since $P = m\dot{v}$, we obtain

$$Pv = mv\dot{v} = \frac{1}{2}\frac{d}{dt}(mv^2).$$

Since $v = \dot{s}$, the left-hand member of the equation represents the rate at which work is being done, and the right-hand member is the rate of increase per unit of time, or the time-flux, of the energy. Hence it follows that the work done in any time gives the change of kinetic energy during that time.

Further it follows that, in a conservative system, that is, in a system in which there is no transformation into heat or other forms of energy, the change in the kinetic energy of the system is entirely due to, and is measured by the work done by the acting forces.

Internal friction, collisions and explosions may produce or destroy energy, but, in what we have called a conservative system, such modes of developing or losing energy are not supposed to exist. In fact, from a mechanical point of view, forces of the character referred to are of the nature of forces extraneous to the system, by means of which the total energy of the system may be increased or diminished.

50. In the chapter of the *Principia* on "Axiomata sive Leges Motus," the concluding paragraph, which is often quoted in support of the Law of Energy as a fundamental law is the following:

Nam si æstimetur agentis actio ex ejus vi et velocitate conjunctim; et similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus et viribus resistendi ab earum attritione, cohæsione, pondere, et acceleratione oriundis; erunt actio et reactio, in omni instrumentorum usu, sibi invicem semper æquales. Et quatenus actio propagatur per instrumentum, et ultimo imprimitur in corpus omne resistens, ejus ultima determinatio determinationi reactionis semper erit contraria.

This passage really shadows forth the principle of energy in its modern form, and indeed states the principle as far as it could be stated at the time when the *Principia* was written.

It was not until Count Rumford began to make observations, and to draw inferences from his observations, followed by a host of other investigators, that the principle of energy presented itself in the form which now renders it the one principle of the greatest utility in the discussion of natural phenomena.

The magnificent intuition, which forms the opening Chapter of the *Mécanique Analytique,* and is the basis of operations of that great work, is, in effect, only a particular case of the general principle of energy.

51. The preceding discussions of this chapter are sufficient for the purposes of the present treatise; but for elaborate accounts, historical and critical, of the laws of motion and the science of energy, the student will consult the *Natural Philosophy* of Sir W. Thomson and Professor Tait.

The student will find valuable expositions of the same ideas, from elementary points of view, in *Matter and Motion* by the late Professor Clerk Maxwell.

Formation of the equations of motion of a particle.

52. We hence see that the solution of any problem on the motion of a particle depends upon the equation

$$P = mf.$$

If x, y, z be the coordinates of a particle referred to three fixed axes, and mX, mY, mZ be the component forces parallel to those axes, then, since the acceleration in any direction is entirely due to the resultant force in that direction, the equations of motion are

$$m\ddot{x} = mX, \quad m\ddot{y} = mY, \quad m\ddot{z} = mZ.$$

If the axes are in motion about the origin, the equations are, dividing by m,

$$\begin{aligned} \dot{u} - v\theta_3 + w\theta_2 &= X \\ \dot{v} - w\theta_1 + u\theta_3 &= Y \\ \dot{w} - u\theta_2 + v\theta_1 &= Z. \end{aligned}$$

If we refer to the tangent, the principal normal and the binormal, the equations are

$$\dot{s} = S, \quad \frac{\dot{s}^2}{\rho} = N, \quad o = T,$$

the forces in the respective directions being, mS, mN, and mT.

If we use cylindrical coordinates, the equations are

$$\ddot{r} - r\dot{\theta}^2 = R,\ \ r\ddot{\theta} + 2\dot{r}\dot{\theta} = T,\ \ \ddot{z} = Z,$$

mR, mT, and mZ being the forces.

And, if we use any other system of representing the accelerations the equations are formed in the same manner.

The integration of these equations, and the determination of the constants of integration by means of the initial circumstances of motion, constitute the solution of the question under discussion.

53. *Equations of motion of a system of particles.*

The inferences which have been drawn from the laws of motion, when expressed in mathematical forms, give the equations of motion of a system.

Thus if m be the mass of a particle of the system, whose coordinates are x, y, z, the rates of change of momenta parallel to the axes, sometimes called effective forces, are $m\ddot{x}$, $m\ddot{y}$, and $m\ddot{z}$.

We have shewn that the system of these quantities is exactly equivalent to the system of acting forces.

If then X, Y, Z, be the sums of the acting forces resolved parallel to the axis, and L, M, N, the sums of the moments of these forces about the axes, we at once obtain

$$\Sigma m\ddot{x} = X,\ \ \Sigma m\ddot{y} = Y,\ \ \Sigma m\ddot{z} = Z,$$

$$\Sigma m(y\ddot{z} - z\ddot{y}) = L,\ \ \Sigma m(z\ddot{x} - x\ddot{z}) = M,\ \ \Sigma m(x\ddot{y} - y\ddot{x}) = N.$$

As in the case of a particle these equations can be presented in various forms, and in a subsequent chapter some of their applications and developments will be considered.

At present they are placed on record, as being immediate consequences of Newton's Laws of Motion.

CHAPTER V.

RECTILINEAR MOTION.

54. THE simplest case of motion is that of a particle in a straight line under the action of forces in that line, and the equation of motion in that case is

$$m\ddot{x} = mX.$$

If the force be constant and equal to mf,

$$\ddot{x} = f \text{ and } \dot{x} = ft + u,$$

u being the initial velocity.

Integrating again, $x = \frac{1}{2}ft^2 + ut + a$, if a be the initial value of x.

The equation may also be written in the form

$$v\frac{dv}{dx} = f,$$

leading to $$v^2 = u^2 + 2fx.$$

55. *Motion of two weights connected by a fine string passing over a smooth fixed pulley.*

If m, m' be the masses, T the tension of the string and x the distance of m from the pulley,

$$m\ddot{x} = mg - T,$$

and similarly $a - x$ being the distance of the other weight,

$$-m'\ddot{x} = m'g - T.$$

Solving these equations we find that

$$\ddot{x} = \frac{m - m'}{m + m'} g, \text{ and } T = \frac{2mm'g}{m + m'}.$$

56. *Motion of a particle, initially at rest, acted upon by a force to a fixed point varying as the distance from that point.*

If the force be $m\mu x$, the equation of motion is

$$\ddot{x} = -\mu x$$

or

$$\ddot{x} + \mu x = 0,$$

the solution of which is

$$x = A \cos \sqrt{\mu} t + B \sin \sqrt{\mu} t,$$

this gives

$$\dot{x} = -A \sqrt{\mu} \sin \sqrt{\mu} t + B \sqrt{\mu} \cos \sqrt{\mu} t.$$

If initially $x = a$, and $\dot{x} = 0$; then $B = 0$,

and

$$x = a \cos \sqrt{\mu} t.$$

The interpretation of this equation is that the particle oscillates through the centre of force between the positions $x = a$ and $x = -a$, the time of a complete oscillation being

$$\frac{2\pi}{\sqrt{\mu}}.$$

If we multiply the equation of motion by $2\dot{x}$ and integrate, we obtain

$$\dot{x}^2 = \mu (a^2 - x^2),$$

shewing as before that the velocity vanishes when $x = \pm a$.

If the force be repulsive the equation of motion is

$$\ddot{x} = \mu x, \text{ or } \ddot{x} - \mu x = 0,$$

leading to

$$x = A\epsilon^{\sqrt{\mu} t} + B\epsilon^{-\sqrt{\mu} t},$$

this gives

$$\dot{x} = \sqrt{\mu} A\epsilon^{\sqrt{\mu} t} - \sqrt{\mu} B\epsilon^{-\sqrt{\mu} t},$$

and introducing the initial conditions we find that

$$a = A + B, \text{ and } O = A - B,$$

so that $$x = \frac{a}{2}(\epsilon^{\sqrt{\mu}t} + \epsilon^{-\sqrt{\mu}t}) = a \cosh \cdot \sqrt{\mu}\,.\,t.$$

As before we can obtain the velocity at once in terms of the distance from the equation, $\dot{x}^2 = \mu(x^2 - a^2)$.

57. *Motion of a particle, initially at rest, under the action of a force varying inversely as the square of the distance from a fixed point.*

In this case, the force being supposed attractive, $\ddot{x} = -\mu x^{-2}$. Multiplying by $2\dot{x}$, and integrating, we obtain

$$\dot{x}^2 = 2\mu(x^{-1} - a^{-1}), \text{ or } \frac{dt}{dx} = \pm \frac{1}{\sqrt{2\mu}}\sqrt{\frac{ax}{a-x}},$$

where a is the initial distance of the particle from the centre of force.

Assuming that the motion is towards the centre of force, we must take the negative sign, and we then have

$$\sqrt{\frac{2\mu}{a}}\frac{dt}{dx} = -\frac{x}{\sqrt{ax - x^2}} = -\frac{x - \frac{a}{2}}{\sqrt{ax - x^2}} - \frac{\frac{a}{2}}{\sqrt{ax - x^2}},$$

and $$\sqrt{\frac{2\mu}{a}}\,.\,t + C = \sqrt{ax - x^2} + \frac{a}{2}\cos^{-1}\frac{2x - a}{a};$$

when $$t = 0,\ x = a, \text{ and } \therefore C = 0,$$

and $$t = \sqrt{\frac{a}{2\mu}}\left\{\sqrt{ax - x^2} + \frac{a}{2}\cos^{-1}\frac{2x - a}{a}\right\}.$$

Putting $x = 0$, we obtain the time from the initial position to the centre of force, which is

$$\frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}.$$

At this point the velocity is infinite, and, as the particle passes through the centre of force, the direction of the force changes, and the motion of the particle is retarded.

Imagine now that the particle is projected away *from* the origin at the initial distance c (less than a) with the velocity

$$\sqrt{2\mu\,(c^{-1}-a^{-1})}.$$

The equation of motion is the same, and we obtain

$$\dot{x}^2 = 2\mu x^{-1} + C.$$

Initially $\qquad x = c$ and $\dot{x} = \sqrt{2\mu\,(c^{-1}-a^{-1})}$;

$\therefore \qquad C = -\,2\mu a^{-1},$

$$\dot{x}^2 = 2\mu\,(x^{-1}-a^{-1}),$$

and $\qquad \dot{x} = \sqrt{2\mu\,(x^{-1}-a^{-1})},$

taking the positive sign as the motion is outwards. This shews that the motion is exactly reversed, and that the particle will traverse the distance from $x = c$ to $x = a$ and then come to rest, the time being exactly the same as in the inward motion from $x = a$ to $x = c$.

This is an instance of Mechanical Reversion, and hence it appears that the particle, in the first case, on arriving at the centre of force, will pass through and repeat its previous motion in exactly the reverse order, and thus perform complete oscillations in the time

$$2\pi\sqrt{\frac{a^3}{2\mu}}.$$

58. We have found that in the case of Art. (56), the time to the centre of force is independent of the initial distance, and, in the case of Art. (57), is proportional to $a^{\frac{3}{2}}$.

The consideration of dimensions enables us to predict each of these results.

In each case, if a be the distance and μ the acceleration at the unit of distance, the time must depend upon a and μ.

In the first case, if we assume that $t \propto a^p\mu^q$, and observe that t is of no dimensions in line, and that, μx being an acceleration, μ is of no dimensions in line, and of -2 dimensions in time, we see that

$$p = 0 \text{ and } 1 = -\,2q,$$

so that $t \propto \dfrac{1}{\sqrt{\mu}}$ and is independent of the distance.

In the second case μ is of three dimensions in line, since $\frac{\mu}{x^2}$ is an acceleration, and therefore

$$p + 3q = 0, \text{ and } 1 = -2q,$$

so that
$$t \propto \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

Conversely, if the time to the centre be independent of the initial distance, and depend only upon μ, and if we assume that the force is proportional to some power of the distance, say the n^{th} power, then, μx^n representing an acceleration, which is of one dimension in line, and μ being a function of the time only, it follows that $n = 1$, so that, if the force vary as some power of the distance, the only possible law is that of the direct distance.

59. *Motion of a particle initially at rest under the attraction of a solid sphere, the particles of which attract according to the law of nature.*

If μ be the mass of the sphere and m the mass of the particle, the force on the particle when outside at a distance r is $m\mu r^{-2}$, and the case is therefore solved in the preceding article.

If however the particle, on arriving at the sphere, be supposed to enter into a fine straight tube, in the line of its motion, passing through the centre, the force at the distance r, less than the radius of the sphere, is

$$\frac{4}{3}\pi\rho r^3 \div r^2, \text{ or } \frac{4}{3}\pi\rho r,$$

ρ being the density.

This being proportional to the distance the motion inside the sphere is determined as in the case of Art. (56).

If the tube extend through the sphere, it will be found that the time of a complete oscillation is

$$4\left\{\sqrt{\frac{b^3}{2\mu}}\cos^{-1}\sqrt{\frac{a}{b}} + \sqrt{\frac{ab(b-a)}{\mu}} + \sqrt{\frac{a^3}{\mu}}\sin^{-1}\sqrt{\frac{b}{3b-2a}}\right\},$$

where a is the radius of the sphere, and b the initial distance of the particle from the centre of the sphere.

60. *Motion in a straight line, in which a centre of force, the attraction to which varies as the distance, is moving with a given constant acceleration f.*

If the original position of the centre of force be taken as the origin, the equation of motion is

$$\ddot{x} = -\mu\left(x - \tfrac{1}{2}ft^2\right),$$

or, if $x - \frac{1}{2}ft^2 = r$,

$$\ddot{r} + f = -\mu r, \text{ or } \ddot{r} + \mu\left(r + \frac{f}{\mu}\right) = 0,$$

the solution of which is

$$r + \frac{f}{\mu} = A \cos\sqrt{\mu}t + B \sin\sqrt{\mu}t.$$

Introducing the initial conditions, we find A and B, and thus determine x in terms of the time.

61. *Motion of a heavy particle, suspended from a fixed point by an elastic string.*

If x be the length at the time t, and T the tension, the equation of motion is

$$m\ddot{x} = mg - T = mg - \lambda\frac{x-a}{a}, \text{ by Hooke's Law,}$$

or
$$\ddot{x} + \frac{\lambda}{ma}\left(x - a - \frac{mag}{\lambda}\right) = 0,$$

whence
$$x - a - \frac{mag}{\lambda} = A \cos\sqrt{\frac{\lambda}{ma}}\,t + B \sin\sqrt{\frac{\lambda}{ma}}\,t.$$

Suppose that initially the particle is held at the distance a, the natural length of the string, and then let go; that is, when $t = 0$, let $x = a$, and $\dot{x} = 0$, then

$$x = a + \frac{mag}{\lambda}\left(1 - \cos\sqrt{\frac{\lambda}{ma}}\,t\right).$$

This shews that the particle descends through the space $2a\frac{mg}{\lambda}$, and then rises again to its initial position and continues to oscillate, the time of a complete oscillation being $2\pi\sqrt{\frac{ma}{\lambda}}$.

The range of oscillation can be obtained at once by the principle of energy, for the particle will fall until the gain of potential energy developed by extension is equal to the loss of potential energy due to the fall.

Now the potential energy of a stretched elastic string

$$= \tfrac{1}{2}\,(\text{Tension})\,(\text{Extension}),$$

and therefore, if z be the total fall,

$$\tfrac{1}{2}\left(\lambda\frac{z}{a}\right)z = mgz, \text{ or } z = 2a\frac{mg}{\lambda}.$$

62. *Fall of a heavy particle in a resisting medium when the force of resistance is proportional to the velocity.*

Measuring x downwards the equation of motion is

$$x = g - k\dot{x},$$

which gives $$\dot{x} + kx = gt,$$

if the particle fall from rest,

and ∴ $$x\epsilon^{kt} = \int gt\epsilon^{kt}\,dt,$$ [Chap. II.]

or $$x = \frac{gt}{k} - \frac{g}{k^2} + \frac{g}{k^2}\epsilon^{-kt},$$

taking $x = 0$, when $t = 0$.

Hence $$\dot{x} = \frac{g}{k} - \frac{g}{k}\epsilon^{-kt},$$

and, if t increase indefinitely, $\dot{x} = \frac{g}{k}$.

This is called the terminal velocity.

63. *Fall of a heavy particle in a medium, the resistance of which varies as the square of the velocity.*

Measuring x downwards the equation of motion is

$$v\frac{dv}{dx}=g-kv^2,\ \text{or}\ \frac{d.v^2}{dx}+2kv^2=2g,$$

from which

$$v^2\epsilon^{2kx}=\frac{g}{k}\epsilon^{2kx}+C,$$

and, choosing the origin so that $v=0$ when $x=0$,

$$v^2=\frac{g}{k}(1-\epsilon^{-2kx}).$$

In this case the terminal velocity is $\sqrt{\frac{g}{k}}$.

Further,

$$\frac{dt}{dx}=\sqrt{\frac{k}{g}}\frac{\epsilon^{kx}}{\sqrt{\epsilon^{2kx}-1}};$$

$$\therefore\ \epsilon^{\sqrt{kg}t}=\epsilon^{kx}+\sqrt{\epsilon^{2kx}-1},$$

from which we obtain,

$$kx=\log\left(\frac{\epsilon^{\sqrt{kg}.t}+\epsilon^{-\sqrt{kg}.t}}{2}\right).$$

Or we may use the equation,

$$\dot{v}=g-kv^2,$$

which leads to

$$\sqrt{\frac{k}{g}}\,\dot{x}=\frac{\epsilon^{2\sqrt{kg}.t}-1}{\epsilon^{2\sqrt{kg}.t}+1}=\frac{\epsilon^{\sqrt{kg}.t}-\epsilon^{-\sqrt{kg}.t}}{\epsilon^{\sqrt{kg}.t}+\epsilon^{-\sqrt{kg}.t}},$$

and gives the same value of x as before.

64. *Motion of a heavy particle projected vertically upwards in the same medium.*

In this case, measuring x upwards,

$$v\frac{dv}{dx}=-g-kv^2.$$

Integrating and taking u as the initial velocity,

$$\frac{g+kv^2}{g+ku^2}=\epsilon^{-2kx},$$

shewing that the particle rises to the height

$$\frac{1}{2k}\log\left(1+\frac{ku^2}{g}\right).$$

Further $$kv^2 = (g + ku^2)\,\epsilon^{-2kx} - g,$$

$$\therefore \sqrt{kg}\,.\,t = \sin^{-1}\frac{\epsilon^{kx}}{\sqrt{1+\frac{ku^2}{g}}} - \sin^{-1}\frac{1}{\sqrt{1+\frac{ku^2}{g}}}$$

which gives

$$\epsilon^{kx} = \sqrt{\frac{k}{g}}\,.\,u\sin(\sqrt{kg}\,.\,t) + \cos(\sqrt{kg}\,.\,t).$$

Or, starting from the equation $\dot{v} = -g - kv^2$, we obtain

$$-\sqrt{gk}\,t = \tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right),$$

and therefore

$$v\sqrt{\frac{k}{g}} = \frac{u\sqrt{\frac{k}{g}} - \tan\sqrt{kg}.t}{1 + u\sqrt{\frac{k}{g}}\tan\sqrt{kg}.t} = \frac{u\sqrt{k}\cos\sqrt{kg}.t - \sqrt{g}\sin\sqrt{kg}.t}{\sqrt{g}\cos\sqrt{kg}\cdot t + u\sqrt{k}\sin\sqrt{kg}.t},$$

giving the same value of x as before.

65. *A particle moves from rest, in a medium the resistance of which varies as the square of the velocity, under the action of a force to a fixed point varying as the distance.*

In this case

$$v\frac{dv}{dx} = kv^2 - \mu x,$$

and therefore

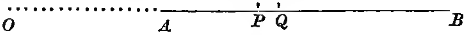
$$v^2 = \frac{\mu x}{k} - \frac{\mu a}{k}\epsilon^{2k(x-a)} + \frac{\mu}{2k^2}(1 - \epsilon^{2k(x-a)}),$$

observing that $v = 0$ when $x = a$.

66. *Motion of a piece of uniform chain in a straight line, under the action of forces in that line.*

Taking a fixed point O in the line, let x be the distance from O of one end A of the chain and take r as the distance from A of a point P of the chain.

The motion of the element PQ (δr) depends upon the tensions at P and Q and the acting force.



If m be the mass of unit length, the mass of the element is $m\delta r$, and if $m\delta rX$ be the force acting upon it, the equation of motion is

$$m\delta r\,.\,\ddot{x} = \delta T + m\delta r\,.\,X,$$

taking T as the tension at P, and observing that the acceleration of every point of the string is the same as that of the point A.

Integrating this equation over the length of the string we shall obtain an equation for determining x in terms of the time.

Suppose for instance that the force is repulsive and varies as the distance from the point O, or that

$$X = \mu\,(x + r).$$

Integrating,

$$mr\ddot{x} = T + m\mu\left(rx + \frac{r^2}{2}\right) + C,$$

and, observing that $T = 0$, when $r = 0$ and when $r = a$,

$$\ddot{x} = \mu\left(x + \frac{a}{2}\right),$$

the solution of which is the same as in previous cases.

Substituting for $\ddot{x}$ we find that

$$T = \tfrac{1}{2}\,m\mu r\,(a - r).$$

67. *Direct impact of elastic balls on each other.*

If two elastic balls impinge directly on each other, that is, if the line joining their centres be the line of motion of each ball, the effect of the impact is an immediate change in the momentum of each ball.

But, since action and reaction are equal and opposite the momentum added to one ball is equal to that which is lost by the other so that the total momentum remains unchanged.

This gives one equation of motion.

For another we appeal to experiment; and assume the experimental law that, if e be the coefficient of elasticity, the relative velocity of the two balls after impact is reversed in direction and is to the relative velocity before impact in the ratio of e to unity.

Hence if the ball m impinge with velocity u on the ball m' moving in the same direction with velocity u', and if v and v' be the velocities after impact, both measured in the same direction as before, we have the equations,

$$mv + m'v' = mu + m'u',$$

$$v' - v = e(u - u'),$$

from which we obtain

$$(m + m')v' = u(m + em) + u'(m' - em),$$

$$(m + m')v = u(m - em') + u'(m' + em').$$

It is worth mentioning that, in all cases of the impact of elastic bodies, energy is lost by impact; if the elasticity be perfect, that is, if $e = 1$, no energy is lost.

If two elastic balls impinge obliquely on each other, all that is necessary is to resolve the velocities parallel and perpendicular to the line of centres; the motions perpendicular to the line of centres are unchanged, and the preceding equations determine the changes of motion along the line of centres.

68. *In the case of Art.* (55), *it is required to examine the effect of suddenly attaching a weight, mass μ, to any point of the ascending string.*

The mass μ, having no momentum before it is attached, acquires momentum instantaneously, and if m be the descending body its motion is suddenly checked, while the portion of string between μ and m' is slackened and m' rises freely.

If u be the velocity with which the two are moving at the instant before μ is attached, and u' immediately afterwards,

$$(m+\mu)\,u' = mu,$$

since the momentum in the direction of motion is unchanged.

The impulsive tension Q of the string is given by the equations

$$m\,(u'-u) = -Q, \quad \mu u' = Q,$$

the effect of the impulse on each body being change of momentum.

Subsequently, if t be the time which elapses before the lower string becomes tightened,

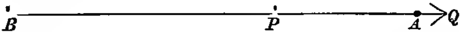
$$ut - \tfrac{1}{2}gt^2 = u't + \tfrac{1}{2}\,\frac{m-\mu}{m+\mu}\,gt^2;$$

this determines t, and therefore determines the velocities of m' and of μ and m at that time.

A jerk then takes place, and the momentum of the system in direction of motion remaining unchanged the new velocity is at once determined and the subsequent acceleration is

$$\frac{m-\mu-m'}{m+\mu+m'}\,g.$$

69. *A straight piece of uniform chain lying on a smooth horizontal table receives at one end a given impulse in direction of its length; it is required to determine the motion and the impulsive tension at any point.*



Let m be the mass of the chain, and a its length; then if v be the velocity produced by the impulse,

$$Q = mv,$$

and, if T be the impulsive tension at a point P,

$$T = m\,\frac{BP}{a}\,v,$$

for the mass of BP is set in motion by the impulse T.

70. *A heavy uniform chain is suspended by one end above a horizontal table, its lower end being just above the table; if it be allowed to fall, it is required to find the pressure on the table.*

We have seen that force is measured by the rate of production, or destruction, of momentum.

As the chain falls, the table receives an infinite number of infinitely small impulses, and the result is that a finite varying pressure is produced, which, added to the weight of the portion coiled up at the instant considered, gives the pressure on the table at that instant.

When a length x has been coiled up, the velocity is $\sqrt{2gx}$, and therefore the portion coiled up in a small time δt is $\delta t \sqrt{2gx}$, and the momentum of this portion, which is destroyed in the time δt,

$$= \frac{M\delta t\sqrt{2gx}}{a} \cdot \sqrt{2gx} = M\frac{2gx}{a}\delta t,$$

M being the mass of the chain and a its length.

Hence momentum is being destroyed at the rate of $2Mg\,\frac{x}{a}$ per unit of time; and therefore, adding the weight of the coil, the pressure on the table is three times the weight of the coil.

71. *One end, B, of a heavy uniform chain hangs over a small pulley A, and the other is coiled up on a table at C; if B preponderate it is required to determine the motion and the tension at C.*

It is easily seen in this case that all internal tensions neutralise each other, and that the momentum of the system in the direction of motion is due to the external forces acting on the system in that direction, that is to gravity, and the reaction of the table.

This reaction is equal to the weight of the coil on the table, and the resulting force in direction of the motion is the difference of the weights of the two straight portions of chain.

If $AC = a$, and $AB = x$, and if v be the velocity, the momentum $= \mu (x + a) v$, μ being the mass of an unit of length; therefore

$$\frac{d}{dt}\{(x + a) v\} = g (x - a),$$

or, since $\frac{dx}{dt} = v$, $(x + a) v \frac{dv}{dx} + v^2 = g (x - a)$,

the integral of this equation is

$$(x + a)^2 v^2 = 2g \left(\frac{x^3}{3} - a^2 x\right) + C,$$

the constant being determined by initial conditions.

If for instance $x = a$, initially, that is, if x be just greater than a, $C = \frac{4}{3} ga^3$,

$$(x + a)^2 v^2 = \frac{2g}{3}(x - a)^2 (x + 2a).$$

Also, (the tension at C) × δt = momentum generated in the time δt by the action of the tension

$$= (\mu v \delta t) v;$$

therefore $\qquad$ tension $= \mu v^2$.

Or, we might have arranged the process thus:

calling the tension T,

$$\mu (x + a) v \frac{dv}{dx} = \mu g (x - a) - T$$

is the equation of motion of the portion of chain CAB, and, as above, $T = \mu v^2$.

72. *A spherical raindrop, as it falls, receives continually by precipitation of vapour, an accession of mass proportional to its surface; neglecting the resistance of the air, it is required to determine its motion.*

If a be the initial radius, and e the thickness of the shell deposited in the unit of time, the radius (r) at the time t

$$= a + et.$$

Hence the momentum at the time t

$$=\tfrac{4}{3}\pi\rho\,(a+et)^3\,v,$$

and $$\frac{d}{dt}\,.\left(\tfrac{4}{3}\pi\rho\,\overline{a+et}|^3\,v\right)=g\,.\,\tfrac{4}{3}\pi\rho\,\overline{a+et}|^3,$$

and therefore $$\frac{dv}{dt}+\frac{3ev}{a+et}=g,$$

and $$v=\frac{g}{4e}\left\{a+et-\frac{a^4}{(a+et)^3}\right\}.$$

This article and the two preceding were published in 1873, in a paper in the *Mathematical Journal* for that year.

73. *Fall of snow down a sloping roof.*

Imagine the snow to be just supported by friction or adhesion, and that a very slight downward impulse is given to the top line of the snow just below the ridge of the roof.

In that case the snow will slide down from the top and gradually set the whole in motion.

Take b as the breadth in motion, and m as the mass of unit area;

then, neglecting the friction on the mass in motion which is practically very slight, the equation of motion is

$$\frac{d}{dt}(mbx\dot{x})=mbxg\sin\alpha,$$

or $$x\ddot{x}+\dot{x}^2=gx\sin\alpha,$$

which gives $$\dot{x}^2=\tfrac{2}{3}\,gx\sin\alpha,$$

and shews that the acceleration is one-third of that of a mass sliding freely.

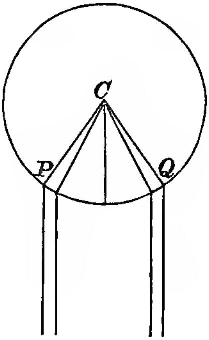
74. *The equilibrium and motion of a heavy ball, supported by a vertical jet of sand.*

We shall consider the case of a cylindrical homogeneous jet, supporting the ball symmetrically, and assume that the reflected particles of sand do not interfere with the ascending particles.

The weight of the ball will be equal to the rate at which momentum is being destroyed, or created in a reversed direction, when resolved vertically.

In other words the weight will be equal to the resultant, which is clearly vertical, of the negative time-fluxes of the momenta.

We shall assume that the velocity of the jet is considerable, so that we may neglect the changes in the velocities of its particles due to the action of gravity.



If m be the mass of the unit of volume of the sand, and u its velocity, the quantity which impinges on an elementary zone in the time δt is

$$m \, . \, 2\pi a \sin\theta \, . \, a\cos\theta\delta\theta \, . \, u\delta t,$$

and the normal component of the momentum of this quantity of sand is

$$2\pi m a^2 u^2 \sin\theta \cos^2\theta\delta\theta\delta t.$$

Multiplying this by $1+e$, where e is the coefficient of elasticity between the ball and the sand, we shall obtain the quantity of motion created in the normal direction outwards round the zone, and, dividing by δt, we then obtain the negative time-flux of the momenta, which is the pressure on the ball.

The last result, when multiplied by $\cos\theta$, will be the resultant vertical pressure on the zone, and, if the breadth of the jet subtend an angle 2α at the centre of the sphere, and M be the mass of the sphere, it follows that

$$Mg = \tfrac{1}{2}\,\pi m a^2 u^2\,(1+e)\,(1-\cos^4\alpha).$$

If the ball be in motion, suppose that, at any instant its vertical velocity is v; then the equation of motion of the ball is

$$M\dot{v} = \tfrac{1}{2}\,\pi m a^2\,(u-v)^2\,(1+e)\,(1-\cos^4\alpha) - Mg,$$

or

$$v\,\frac{dv}{dx} = -\frac{g}{u^2}\,(2uv - v^2).$$

Integrating we obtain

$$2u - v = Ce^{\frac{gx}{u^2}};$$

and, if we suppose the ball to start upwards with a velocity v', it will have its velocity destroyed after ascending through the space

$$x = \frac{u^2}{g}\log\frac{2u}{2u-v'},$$

and will then be under conditions consistent with equilibrium.

The time in which this takes place is obtained by integrating the equation

$$\dot{v} = -\frac{g}{u^2}\,(2uv - v^2),$$

and the theoretical result is that an infinite time must elapse before the ball absolutely loses its velocity.

The simplest method of illustrating the idea of this article is to employ a jet of water. The ball rises and falls intermittently, and is occasionally at rest for a sensible time. As in the case imagined of a jet of sand, the pressure is due to the rate of destruction, and of creation in the contrary direction, of the momenta, in directions perpendicular to the surface of the sphere, of all the elementary cylindrical shells which constitute the impinging stream.

EXAMPLES.

1. A smooth wedge on a horizontal plane is moved from rest with an uniform acceleration; find the direction and amount of the acceleration that a heavy particle placed on its inclined plane surface may be in equilibrium relative to it.

If the acceleration be given, find the motion of the particle, supposed initially at rest, upon the inclined surface.

2. Particles slide from a fixed point down rough planes to points in the surface of a cone, whose axis, passing in direction through the point, is vertical, and vertex upwards. Shew that, if the vertical angle of the cone $= 2 \tan^{-1} \frac{1}{\mu}$, the particles will all have the same velocity on arriving at the cone.

3. Give a geometrical construction for determining the straight line of quickest descent of a heavy particle from a given point to a given curve.

If the curve be a conic, with its vertex upwards, the length of the line of quickest descent from the focus to the curve is equal to the latus rectum.

4. Determine the motion of a particle, initially at rest, under the action of a force to a fixed point varying inversely as the cube of the distance.

5. A train goes from one station to another a mile off, being uniformly accelerated from rest in the first two-thirds of the distance, and being brought to rest by uniform retardation in the remaining one-third of the distance, and taking 3 minutes to perform the journey. Find the acceleration and retardation and the maximum velocity acquired.

6. A hyperbola is placed in a vertical plane with its transverse axis horizontal; prove that when the time of descent down a diameter is least, the conjugate diameter is equal to the distance between the foci.

7. Find the locus of points from which inelastic particles may be let fall on a smooth inclined plane, so as always to have the same velocity on arriving at the same horizontal line in the plane.

8. Two equal weights are fastened to the extremities of a string and are then hung over two small smooth pullies *A*, *B* which are in the same horizontal line. If a third equal weight be fastened at the middle of the horizontal portion *AB* of the string, shew that it will descend a distance equal to two-thirds of *AB*, and find the velocity in any position.

9. Two bodies, $2P$ and P, are connected by an inextensible string, which passes over a fixed smooth pulley; determine the motion and the tension of the string.

If, after the motion has gone on for one second, another body P, having no velocity, be suddenly attached to the descending body $2P$, determine completely the subsequent motion of the system.

10. Two elastic balls impinge obliquely, so that their directions of motion are interchanged: prove that the product of their velocities is unaltered, and that if their masses be equal their directions make equal angles with the line of impact and that their velocities are unaltered.

11. In an Atwood's machine, if the string can only bear a strain of one-fifth of the sum of the weights at its two ends, shew that the larger weight is not less than 13 times the smaller weight, and that the least acceleration possible is $\sqrt{\frac{3}{5}}g$.

12. If two elastic balls impinge on each other with equal and opposite momenta, their kinetic energy will be reduced in the ratio $e^2 : 1$.

13. A particle is projected in a resisting medium towards a centre of attractive force which varies as the inverse cube of the distance: the resistance of the medium varies as its density and as the square of the velocity, the density varying as the inverse cube of the distance from the

centre: prove that the particle's velocity on reaching the centre is independent both of its initial distance and of its initial velocity.

14. A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left(1-\frac{a}{x}\right)$ towards the fixed point when x is greater than a, but $\mu\left(\frac{a^2}{x^2}-\frac{a}{x}\right)$ from the same point when x is less than a: prove that the particle will oscillate through a space $\frac{b^2-a^2}{b}$.

15. In a single moveable pulley when there is equilibrium the power and the weight hang by vertical strings; the weight being doubled and the power being halved, prove that the tension of the string will be unchanged.

16. If in the second system of pullies there are n strings at the lower block, prove that the upward acceleration of W due to a power P will be

$$\frac{nP-W}{n^2P+W}\cdot g.$$

If when W has an upward velocity v, the weight P reach the ground, prove that there will presently be upon the string an impulsive strain

$$\frac{nPWv}{n^2P+W}.$$

17. A series of n elastic spheres whose masses are 1, e, e^2, &c. are at rest, separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Prove that the final kinetic energy of the system is $\frac{1}{2}(1-e+e^n)\,u^2$.

18. Two weights P and Q, connected by a string passing over a smooth pulley, are held at a distance c above a hard inelastic horizontal plane and let go. After a series of

impacts by the heavier weight P on the plane, the system at length comes to rest. Shew that the whole time of motion is

$$3\sqrt{2}\sqrt{\frac{P+Q}{P-Q}}\cdot\frac{c}{g}.$$

19. Two balls, of elasticity e, moving in parallel directions with equal momenta, impinge; prove that, if their directions of motion be opposite, they will move after impact in parallel directions with equal momenta; and that these directions will be perpendicular to the original direction if their common normal is inclined at an angle $\sec^{-1}(1+e)$ to that direction.

20. Prove that, in order to produce the greatest deviation in the direction of a smooth billiard ball of diameter a by impact on another equal ball at rest, the former must be projected in a direction making an angle

$$\sin^{-1}\frac{a}{c}\sqrt{\frac{1-e}{3-e}}$$

with the line (of length c) joining the two centres; e being the coefficient of elasticity.

21. A ball A strikes directly a ball B at rest, B strikes directly C at rest and is then again struck by A; if A is reduced to rest and B and C move with equal velocities, prove that

$$A : B : C :: 3 : 1 : 2,$$

all the balls being perfectly elastic.

22. Three balls of masses A, B, C are at rest with their centres in a straight line; if A be projected towards B, shew that there will be four impacts if $\dfrac{B(A+B+C)}{AC}$ be $< \dfrac{1+e^2}{2e}$, and only three if it be $> \dfrac{1+e^2}{2e}$ and $< \dfrac{1+e+e^2}{e}$, and only two if it be $> \dfrac{1+e+e^2}{e}$.

23. A weight P hanging vertically just supports a weight W in that system of pullies in which there is only one string. Shew that, neglecting the masses of the pullies, if P and W be interchanged their centre of gravity will descend with an acceleration

$$\frac{(W-P)^2}{W^2-WP+P^2}g.$$

24. If the weight (P), on a wheel and axle, suspended from the wheel preponderate over the weight (W) suspended from the axle, prove that the acceleration of P is

$$g\frac{a^2P-abW}{b^2W+a^2P},$$

where a and b are the radii of the wheel and axle, the inertia of the wheel and axle being neglected.

If an additional weight (w) be suddenly attached to W, find the impulsive tensions of the two strings.

25. Two trains of equal weight are being drawn along smooth level rails by engines, one of which exerts a constant tractive force, while the other's rate of working is uniform. Prove that if their velocities at two instants are equal, the second train moves through the greater distance during the interval between the two instants, and that they are working at the same rate, at the end of half this interval.

26. Two given points are in the same vertical line; shew that the locus of the points in a vertical plane through them, from which the times of descent to the two points are the same, is a rectangular hyperbola.

Shew also that, if two equal circles be in the same vertical plane, the locus of the points from which the times of shortest descent to the circles are the same is a rectangular hyperbola.

27. A particle moves in a straight line under a centre of attractive force μ^2r in that straight line; if it be initially at a distance c from the centre of force and be projected in the

straight line with velocity V, shew that it will arrive at the origin in a time equal to

$$\frac{1}{\mu}\sin^{-1}\frac{\mu c}{\sqrt{V^2+\mu^2c^2}}.$$

28. A cycloid has its base horizontal and vertex upwards; prove that the time of falling down any radius of curvature is constant.

29. The mean time of descent down a given inclined plane of unknown roughness is equal to twice that down an equal smooth plane, all coefficients of friction, for which motion is possible, being considered equally probable.

30. A heavy particle is attached by an elastic string to a fixed point on a smooth horizontal table; the particle is drawn out along the table till the string is double its natural length (a) and it is then let go; find the velocity of the particle in any position and shew that it will return to the starting point after a time $= 2\sqrt{\frac{a}{g}}(\pi+2)$: the modulus of elasticity of the string being equal to the weight of the particle.

31. Two equal particles which mutually repel one another with a force varying as the distance between them are connected by a light elastic string; find the condition that the motion may be oscillatory; and assuming that the particles would rest in equilibrium with the string stretched to twice its natural length find the amplitude of the oscillation if the particles just meet.

32. Two particles start simultaneously from the same point and move along two straight lines, the one with uniform velocity, and the other from rest with uniform acceleration. Prove that the line joining the particles at any time is always a tangent to a fixed parabola.

33. An infinite number of particles are arranged along a curve; they move normally to the curve with velocities which are always proportional to the perpendicular from the

origin on the tangent to the locus at any instant. Prove that they will always lie in a similar curve with the origin for a centre of similitude; and that if they move so as to approach the origin, they will reach it together after an infinite time.

34. *ABCD* is a smooth tube in a vertical plane bent upwards at *B* and *C*, *BC* being horizontal, and the angles *ABC*, *BCD*, being equal and obtuse. An inelastic particle slides from a given point in *AB*. Find the time it will take to reach its highest position in *CD*; and if this time be equal to that of the next similar stage of motion, and angle $ABC = 135°$, shew that the original vertical height of the particle above $BC = \frac{2}{3} BC$.

35. A smooth circular hoop, radius a, rests on a smooth horizontal table; a small spherical mass is projected in any direction from the centre of the circle with velocity v: prove that the whole time that elapses until the nth impact is

$$\frac{a}{v} \cdot \frac{2 - e^{n-1} - e^n}{e^{n-1} - e^n},$$

where e is the coefficient of elasticity.

Also find the ultimate velocity of the ring and sphere after an infinite time.

36. An elastic string is extended between two fixed points to double its natural length, and a particle of mass m is fastened to the middle point of the string. If the particle be drawn towards one of the fixed points through half its distance from that point, and then let go, find the greatest velocity which it subsequently acquires.

If a be the natural length of the string, prove that the time of a complete oscillation is $\pi\sqrt{ma} \div \sqrt{\lambda}$.

37. A particle is placed initially at a distance a from a centre of force the attraction to which varies inversely as the distance; prove that the time of arriving at the centre of force is $a\sqrt{\dfrac{\pi}{2\mu}}$.

38. The upper extremity of a piece of chain, hanging vertically is made to move upwards, with a given acceleration; find the tension at any point of the chain.

39. An endless elastic string, modulus λ and natural length $2\pi c$, is placed in the form of a circle on a smooth horizontal plane, and is acted upon by a force from its centre equal to μr per unit mass of the string. Shew that its radius will vary harmonically about a mean length

$$2\pi\lambda c \div 2\pi\lambda - m\mu c,$$

m being the mass of the string, if $2\pi\lambda > m\mu c$.

Examine the case when $2\pi\lambda = m\mu c$.

40. A particle is placed within a straight tube of length a and small section and the ends of the tube closed. If the tube is placed on a smooth horizontal plane and projected in the direction of its length, prove that the distance advanced by the tube between the 1st and $(n+1)^{\text{th}}$ impacts between the particle and the ends of the tube is $\frac{a}{2}(e^{-n}-1)$; the masses of the tube and particle being equal and the coefficient of restitution between the particle and the ends of the tube being e.

41. Two equally elastic balls, not in the same vertical line, are dropped upon a hard horizontal plane: if the balls ever come again simultaneously to positions of zero velocity, prove that their initial heights above the plane are as

$$(e^m - 1)^2 \text{ to } (e^n - 1)^2,$$

where m and n are positive integers, and e is the modulus of elasticity.

42. P is a point to which the time t of sliding from A and B in straight lines is the same. Find another point Q for which the times from A and B are also t; and shew that if t' be the time from P to Q or Q to P

$$\frac{t^2 \sim t'^2}{t^2 + t'^2} = \frac{AP\,.\,BP}{AQ\,.\,BQ} \text{ or } \frac{AQ\,.\,BQ}{AP\,.\,BP}:$$

43. A particle is projected with a given velocity in a medium in which the resistance varies as the cube of the velocity; find the time in which it will traverse a given distance, and the velocity which it will have at the end of a given time.

44. A string hangs over a fixed pulley; a weight of two pounds hangs at one end, and a pulley at the other; over the pulley hangs a string, carrying a weight of one pound at each end; when the whole is in equilibrium, any force is applied to one of the smaller weights: shew that, when it has pulled it down three inches, the other one pound weight, and the two pound weight has each risen one inch; shew also that, if the motion of the weight to which the force was applied be stopped in any gradual manner, the whole will be brought to rest, and the distances travelled by the weights will be as 3 : 1 : 1.

45. Two inclined planes of the same altitude and inclinations α, β, are placed back to back with an interstice between. Two weights P, Q are placed one on each plane at the bottom and connected by a string which passes over two small smooth pullies at the top and under a moveable pulley, weight W, which hangs between the two planes, the free portions of the string being parallel. Find the least value of W in order that both weights may be drawn up: and if they arrive at the top at the same time, prove that

$$\frac{4(\sin^2\alpha - \sin^2\beta)}{W}$$

$$= \frac{2\sin\alpha + \sin\alpha\sin\beta + \sin^2\beta}{P} - \frac{2\sin\beta + \sin\alpha\sin\beta + \sin^2\alpha}{Q}.$$

46. A particle, of mass m, initially at rest at a distance a from the origin, is acted upon by the force $m\mu\left(r + \frac{a^4}{r^3}\right)$ to the origin, r being the distance; find the time in which it arrives at the origin.

47. Examine the motion of a particle, initially at rest in a medium the resistance of which varies as the velocity, under the action of a force to a fixed point varying as the distance.

48. A train travels at the rate of 45 miles an hour. Rain is falling vertically, but owing to the motion of the train, the drops appear to fall past the window at an angle $\tan^{-1} 1{\cdot}5$ with the vertical. Find the velocity of the raindrops.

If the raindrop were divided into 1000^6 equal spherical portions, prove that the cloud so formed would have a velocity of ·044 ft. per second.

The velocities are supposed to be full speed velocities, and the resistance of the air to vary as the square of the product of the velocity into the diameter of the drop.

49. Two imperfectly elastic balls, equal in size, but unequal in mass, are placed between two perfectly hard parallel vertical planes, to which the line joining the centres of the balls is perpendicular, each ball being initially at a distance from the plane nearest to it, inversely proportional to its mass. The balls approach each other, with velocities inversely proportional to their masses; prove that every impact will take place at the same point as the first does.

50. In an Atwood's machine the heavier body P is perfectly elastic, and Q is perfectly inelastic, and they start from rest at the same distance a above a fixed horizontal plane; and when P impinges on the plane and rebounds with unchanged velocity, Q strikes against a fixed obstacle and is reduced to instantaneous rest: determine the subsequent motion, and shew that the two bodies are again at instantaneous rest when P is at a height $P^2a \div (P+Q)^2$ above the horizontal plane.

51. A piece of uniform chain hangs vertically from its upper end, with its lower end just above a smooth inclined plane; if it be let go, find the pressure on the plane during the fall.

52. A heavy chain is suspended from one end above a rigid horizontal plane; on the other side of the plane in the vertical line of the chain, there is a centre of force the attraction to which varies inversely as the square of the distance; find the motion of the chain, and the pressure on the table during the fall of the chain upon it.

53. The potential at the distance r from a fixed centre being μr^n, find the motion of a particle originally at rest at a distance a from the centre, and prove that the time of oscillation is

$$\frac{4}{na^{\frac{n-2}{2}}}\cdot\sqrt{\frac{\pi}{2\mu}}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}.$$

54. An elastic flexible ring of natural radius a is stretched upon a circular cylinder whose radius is $\left(1+\sqrt{\frac{2}{e}}\right)a$, and is then pushed off so that all the points of the ring quit the cylinder at once. Assume the law of compression to be that the compressed length is to the natural length as $\lambda : P+\lambda$ where λ is the pressure which would halve the length of the string, and is equal to the tension which would double it according to Hooke's law. Prove that in the subsequent motion the least radius of the ring will be $\frac{a}{e}$, where e is the number 2·718281828.

55. A rope passes over a smooth pulley, having a weight attached to one end, and a monkey hanging at the other end, just balancing the weight. The monkey suddenly starts off, runs up a certain length of the rope at a uniform rate, and then holds on; determine the whole motion, and prove that, if the monkey exert his whole strength in climbing, and be at all fatigued by the effort, he will be certainly jerked off.

56. An uniform string, whose length is l and weight per unit of length w, hangs over a small smooth pulley with its ends just in contact with a horizontal plane; if the string be slightly displaced, shew that when one end has risen through a height h the pressure of the string on the plane is

$$w\left(2l\log\frac{l}{l-h}-h\right),$$

and its resultant pressure on the pulley is

$$wl\,\frac{l-2h}{l-h}.$$

57. Two particles of equal mass m are placed in two smooth straight tubes, between which the shortest distance is c and the angle 2α.

The accelerating effect of the attraction of either upon the other is $mf(\rho)$ at distance ρ. Each particle is initially at rest, one at the foot of the perpendicular, the other at a very small distance β from it, shew that their respective distances from it at the time t will be

$$\tfrac{1}{2}\beta\,\{\cos(t\sqrt{\mu}\sin\alpha)\mp\cos(t\sqrt{\mu}\cos\alpha)\},$$

where $2\mu c = mf(c)$.

58. A ball is supported by a uniform jet of sand, in the form of a thin conical surface, impinging upon it symmetrically as regards its vertical diameter: prove that the weight of the ball is $Qv(1+e)\cos\alpha\cos(\alpha+\beta)$; where Q is the quantity of sand discharged from the jet per unit of time, v the velocity of discharge, e the coefficient of elasticity between the ball and the sand, α the angular radius of the circle in which the jet impinges, β the semi-vertical angle of the jet.

59. Two equal buckets are connected by a string without weight passing over a smooth pulley, and over one of the buckets a heavy chain is held by its upper end, with its lower end just above the base of the bucket; if the upper end be let go, prove that the equilibrium may be maintained by pouring water gently and uniformly into the other bucket, provided the weight of water which can be poured in is three times the weight of the chain. After the chain has entirely fallen in, find its pressure on the bucket in which it lies, supposing the flow of water then to cease.

60. One end of a heavy chain length $3a$ is fastened to a small smooth ring through which the chain is passed so as to be in equilibrium with a length a hanging freely. Prove that if the free end be slightly displaced downwards its velocity V when the length of the free portion is x is given by the equation

$$V^2 = 2g\,\frac{(x-a)^2(x+5a)}{(x+3a)^2}\,.$$

61. A heavy chain, of length $4a$, is coiled up on a horizontal table, at the distance a from one edge of the table, and one end of the chain is then drawn out at right angles to the edge and just over it; the height of the table above the floor being a, investigate completely the motion of the chain.

62. A chain of given length is at rest on a smooth horizontal plane, with one end fastened to a point on the plane, under the action of a repulsive force from that point varying as the distance. If the chain be set free, find the initial change of tension at any point, and the subsequent motion of the chain.

If the chain impinge upon a vertical wall perpendicular to its own direction, find the pressure upon the wall at any subsequent period.

63. On a certain day between one and four o'clock in the afternoon $\frac{1}{2}$ an inch of rain fell. Assuming that the drops were indefinitely small and that their terminal velocity was 10 feet per second, find the impulsive pressure in tons per square mile consequent on the rain-drops being reduced to rest, assuming that a cubic foot of water contains 1000 oz. and that the rain fell uniformly and continuously.

64. A fine string without weight passing over a smooth pulley, supports two equal scale-pans; a heavy chain is held by its upper end above one of the scale-pans, its lower end being just above the scale-pan; if the upper end be let go, determine the motion completely, and find, at any time, the pressure on the scale-pan.

65. A meteor is seen to fall vertically to the Earth, leaving a bright trail behind it. If the resistance of the air be constant and equal to R, the rate of loss of matter burnt off be $\frac{1}{k}Rv$, and if M, V be its mass and velocity just after entering the atmosphere, shew that the velocity after falling through a distance z is given by

$$v^2 = \frac{\lambda^2 \{V^2 + \frac{2}{3}\lambda g - k\}}{(z-\lambda)^2} + \frac{2}{3}(z-\lambda)g + k,$$

where $\lambda = \frac{kM}{R}$.

66. An indefinite quantity of a uniform string is coiled in a heap on the floor of a room and escapes into the room below through a hole in the floor; shew that the velocity of escape can never exceed $\sqrt{ga}$, where a is the height of the hole above the floor of the room below.

67. A chain whose density varies as the distance from the end A is coiled up close to the edge of a smooth table and the end A allowed to hang over. Shew that the motion is uniformly accelerated and the tension at the edge of the table varies as the fourth power of the time elapsed since the commencement of motion.

68. A pulley is fixed above a horizontal plane. Over the pulley passes a fine string which has two equal chains fastened to its two ends. In the position of equilibrium a length c of each chain is vertical, the remainder of the chains being coiled up on the table.

If now one chain be drawn through a distance nc and then let go, prove that the system will next come to rest when the upper end of the other string is at a distance mc below its mean position, m being given by the equation

$$(1-m)\,\epsilon^{m} = (1+n)\epsilon^{-n}.$$

69. A flexible chain hangs in equilibrium over a smooth vertical circle with one end fixed to the extremity of a horizontal diameter and a portion hanging vertically at both sides of the circle; if the fixed end be set free, shew that the equation for determining (y) the distance of the lowest point of the chain from the horizontal diameter during the first part of the motion is

$$\left(l-y+\frac{gt^2}{2}\right)\frac{d^2y}{dt^2}-\left(\frac{dy}{dt}-gt\right)^2=g\left(y+\frac{c}{2}\right)$$

where l is the length of the whole string and $2c$ is the circumference of the circle.

CHAPTER VI.

ACCELERATIONS PARALLEL TO CO-ORDINATE AXES.

75. WE now proceed to illustrate the use of the equations

$$m\ddot{x} = mX, \quad m\ddot{y} = mY,$$

by the consideration of some cases to which these equations are the most easily applied.

If the forces are given, and the initial circumstances of motion, the equations determine the path of the particle; and, if the path be given, with some other condition besides, the forces are determined.

76. *Motion of a heavy particle in a vertical plane.*

Measuring y vertically upwards the equations of motion are

$$\ddot{x} = 0, \quad \ddot{y} = -g.$$

If u be the initial velocity and α the inclination to the horizontal plane of its direction, we obtain

$$\dot{x} = u \cos \alpha, \quad \dot{y} = u \sin \alpha - gt,$$

and if the point of projection be the origin,

$$x = ut \cos \alpha, \quad y = ut \sin \alpha - \tfrac{1}{2}gt^2.$$

Eliminating t we obtain the equation to the path of the particle,

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha},$$

which is a parabola, having its axis vertical and vertex upwards, the co-ordinates of the vertex being

$$u^2 \sin 2\alpha/2g \text{ and } u^2 \sin^2 \alpha/2g.$$

The latus rectum is $2u^2 \cos^2 \alpha/g$, and the height of the directrix above the point of projection is $u^2/2g$.

The direction of motion at the time t is given by the equation

$$\tan\theta = \frac{u \sin\alpha - gt}{u\cos\alpha},$$

and, if v be the velocity at the point (x, y),

$$v^2 = \dot{x}^2 + \dot{y}^2 = u^2 + g^2t^2 - 2ugt \sin\alpha = u^2 - 2gy,$$

shewing that the velocity at any point is equal to the velocity due to a fall to that point from the directrix.

The range of a projectile on the horizontal plane through the point of projection is at once seen to be

$$u^2 \sin 2\alpha/g.$$

To obtain the range on an inclined plane, perpendicular to the plane of motion, through the point of projection, measure x and y parallel and perpendicular to the plane.

Then, if β be the inclination of the plane,

$$\ddot{x} = -g\sin\beta,\ \ddot{y} = -g\cos\beta,$$

$$\therefore\quad x = u\cos(\alpha - \beta) \,.\, t - \tfrac{1}{2}g\sin\beta t^2,$$

$$y = u\sin(\alpha - \beta) \,.\, t - \tfrac{1}{2}g\cos\beta t^2.$$

Putting $y = 0$, and eliminating t, we obtain

$$x = \frac{2u^2 \cos\alpha \sin\overline{\alpha - \beta}}{g\cos^2\beta},$$

which is the range on the inclined plane, and is a maximum when $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \beta\right)$, that is, when the direction of projection bisects the angle between the inclined plane and the vertical.

To find the direction of projection in order that the particle may pass through a given point is the same thing as finding the direction of projection in order that the range on a given inclined plane may be a given quantity.

Hence from above $\cos\alpha\sin\overline{\alpha-\beta}$ is given,

and therefore $\sin\overline{2x-\beta}$ is given.

If then α' and α'' be the two values of α, the expressions $2(\alpha'-\beta)+\beta$ and $2(\alpha''-\beta)+\beta$ are supplementary, and therefore

$$(\alpha'-\beta)+(\alpha''-\beta)=2\left\{\frac{\pi}{4}-\frac{\beta}{2}\right\},$$

shewing that there are two directions of projection and that they are equally inclined to the direction of projection which gives the greatest range.

77. Conversely, if the given path be a parabola, and it be given that the force is parallel to its axis, we have

$$y^2=4ax,$$

with the equations $m\ddot{x}=mX,\ \ddot{y}=0,$

so that $\dot{y}=c$, a constant,

and therefore $\dot{x}=\dfrac{yc}{2a}$ and $\ddot{x}=\dfrac{c^2}{2a}$,

shewing that the force is constant.

78. To find the force perpendicular to the axis under the action of which a conic section can be described, we have the equations,

$$y^2=2lx-nx^2,$$

$$\ddot{x}=0,\ m\ddot{y}=mY.$$

From these we obtain

$$\dot{x}=c,\ \text{and}\ y\dot{y}=(l-nx)c,$$

hence $y\ddot{y}+\dot{y}^2=-nc^2$, and therefore $\ddot{y}=-\dfrac{l^2c^2}{y^3}$,

shewing that the force varies inversely as the cube of the ordinate.

Conversely, if it be given that the force is perpendicular to the axis of x and inversely proportional to y^3, the equations of motion are $\ddot{x}=0,\ \ddot{y}=-\dfrac{\mu}{y^3}$,

and therefore $\dot{x}=c,\ \dot{y}^2=\mu\left(\dfrac{1}{y^2}\pm\dfrac{1}{b^2}\right)$,

c and b being constants.

Hence, $$\frac{1}{c}\frac{dx}{dy} = \frac{by}{\sqrt{\mu(b^2 \pm y^2)}},$$

and therefore $$\frac{x}{c} + a = \frac{b}{\sqrt{\mu}}\sqrt{b^2 \pm y^2},$$

which is the equation of a conic section.

79. *It is required to find the force perpendicular, to the asymptote, under the action of which the curve,* $x^3 + y^3 = a^3$, *can be described.*

In this case, if P be the force,

$$m\ddot{x} = \frac{P}{\sqrt{2}}, \text{ and } m\ddot{y} = \frac{P}{\sqrt{2}}.$$

Hence $$\dot{x} - \dot{y} = c,$$

and from the equation of the curve $x^2\dot{x} + y^2\dot{y} = 0$.

These equations give

$$\dot{x} = \frac{cy^2}{x^2 + y^2}, \quad \dot{y} = \frac{-cx^2}{x^2 + y^2},$$

and $$\ddot{x} = -\frac{2c^2a^3xy}{(x^2 + y^2)^3}, \text{ whence the force;}$$

and the velocity at any point $= \dfrac{c\sqrt{x^4 + y^4}}{x^2 + y^2}$.

80. In all cases in which force acts in parallel lines, the velocity in the direction perpendicular to the force remains constant, and therefore the time of traversing any arc of the orbit is obtained by observing the space, passed over in the direction perpendicular to the force, and dividing this space by the constant velocity.

81. *Motion of a particle under the action of a force to a fixed point proportional to the distance from that point.*

If r be the distance, and μr the force on a particle of unit mass, the equations of motion are

$$\ddot{x} = -\mu r \cos\theta = -\mu x, \quad \ddot{y} = -\mu r \sin\theta = -\mu y.$$

The integrals of these equations are, Chapter II.

$$x = A \cos\sqrt{\mu}t + B \sin\sqrt{\mu}t, \quad y = C \cos\sqrt{\mu}t + D \sin\sqrt{\mu}t,$$

the constants being determined by the initial circumstances of the motion.

Eliminating t we obtain the path

$$(Cx - Ay)^2 + (By - Dx)^2 = (BC - AD)^2,$$

which is an ellipse.

82. In this case we may usefully employ oblique axes.

O being the centre of force, and A the point of projection, take OA as the axis of x, and the line through O parallel to the direction of projection as the axis of y.

If f, f' be the component accelerations parallel to the axes

$$f \sin\alpha = \frac{d^2}{dt^2}(x \sin\alpha), \text{ and } f' \sin\alpha = \frac{d^2}{dt^2}(y \sin\alpha),$$

so that $$f = \ddot{x} \text{ and } f' = \ddot{y}.$$

The component forces are $-\mu x$ and $-\mu y$, and resolving perpendicular to the axes, we obtain

$$\ddot{x} \sin\alpha = -\mu x \sin\alpha, \quad \ddot{y} \sin\alpha = -\mu y \sin\alpha,$$

or $$\ddot{x} = -\mu x, \quad \ddot{y} = -\mu y.$$

Initially $$x = a, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = v,$$

and therefore $$x = a \cos\sqrt{\mu}t, \quad y = \frac{v}{\sqrt{\mu}} \sin\sqrt{\mu}t,$$

and the path is $$\frac{x^2}{a^2} + \frac{\mu y^2}{v^2} = 1,$$

an ellipse of which $2a$ and $\frac{2v}{\sqrt{\mu}}$ are conjugate diameters.

If the force be repulsive the equations of motion, employing oblique axes, are

$$\ddot{x} = \mu x, \quad \ddot{y} = \mu y,$$

and therefore

$$x = A\epsilon^{\sqrt{\mu}t} + B\epsilon^{-\sqrt{\mu}t}, \quad y = C\epsilon^{\sqrt{\mu}t} + D\epsilon^{-\sqrt{\mu}t},$$

and, with the same initial conditions,

$$x = \frac{a}{2}(\epsilon^{\sqrt{\mu}t} + \epsilon^{-\sqrt{\mu}t}), \quad y = \frac{v}{2\sqrt{\mu}}(\epsilon^{\sqrt{\mu}t} - \epsilon^{-\sqrt{\mu}t}),$$

and the equation of the path is

$$\frac{x^2}{a^2} - \frac{\mu y^2}{v^2} = 1,$$

a hyperbola, of which $2a$ and $\frac{2v}{\sqrt{\mu}}$ are conjugate diameters.

In each case if $2a$ and $2b$ be the conjugate diameters

$$v = \sqrt{\mu} \, . \, b,$$

and, as any point P of the path may be regarded as the initial point, the velocity at P is equal to $\sqrt{\mu} \, . \, CD$, where CD is conjugate to CP.

83. *General deductions from the equations of motion.*

Taking $\ddot{x} = X$, and $\ddot{y} = Y$,

we obtain $x\ddot{y} - y\ddot{x} = xY - yX$,

or $\frac{d}{dt}(x\dot{y} - y\dot{x}) = N$,

if mN be the moment of the forces about the origin.

The angular momentum of the particle is the moment of its momentum, which is

$$m(x\dot{y} - y\dot{x}).$$

If then we denote by mh the angular momentum, we obtain

$$m\dot{h} = mN,$$

that is, the time flux of the angular momentum is equal to the moment of the forces.

Again we know that, if A be the area swept over by the radius vector,

$$2\dot{A} = x\dot{y} - y\dot{x},$$

and $\therefore$ $2\ddot{A} = N$.

If N be zero, that is, if the force be central, $\dot{A}$ is constant, so that the area swept over is in that case proportional to the time.

This is Newton's Proposition I.

Further we obtain, multiplying by $2\dot{x}$, $2\dot{y}$, adding, and integrating,

$$\dot{x}^2 + \dot{y}^2 = 2\int (Xdx + Ydy),$$

or
$$\tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) = \int (Xdx + Ydy).$$

The left-hand side is the kinetic energy of the particle, and the right-hand side is the work done by the force, so that we here have the principle of energy, for a single particle in a field of force, deduced from the equations of motion.

84. *Motion of a heavy particle in medium the resistance of which varies as the velocity.*

Measuring x and y horizontally and vertically from the point of projection, and taking mkv as the force of resistance the equations of motion are

$$\ddot{x} = -k\dot{s}\frac{dx}{ds} = -k\dot{x}, \text{ and } \ddot{y} = -k\dot{s}\frac{dy}{ds} - g = -k\dot{y} - g,$$

and therefore, if u and v be the initial components of the velocity,

$$\dot{x} + kx = u, \quad \dot{y} + ky = v - gt,$$

leading to
$$x = \frac{u}{k}(1 - \epsilon^{-kt}),$$

$$y = \frac{v}{k} - \frac{gt}{k} + \frac{g}{k^2} - \left(\frac{v}{k} + \frac{g}{k^2}\right)\epsilon^{-kt}.$$

85. *Motion of a particle in the same medium under the action of a force to a fixed point varying as the distance from that point.*

The equations of motion are

$$\ddot{x} + k\dot{x} + \mu x = 0, \quad \ddot{y} + k\dot{y} + \mu y = 0,$$

and these are integrated as in Chapter II.

If, initially, $x = a$, $y = 0$, $\dot{x} = 0$, and $\dot{y} = v$, it can be shewn that

$$\frac{x^2}{a^2} + \frac{kxy}{av} + \frac{\mu y^2}{v^2} = \epsilon^{-kt},$$

and $$\tan nt = \frac{2any}{2vx - aky}, \text{ where } n = \sqrt{\mu - \frac{k^2}{4}}.$$

86. *Motion of a heavy particle in a medium the resistance of which varies as the square of the velocity.*

The equations of motion, if mkv^2 be the force of resistance, are

$$\ddot{x} = -kv^2 \frac{dx}{ds} = -k\dot{x}\dot{s}, \quad \ddot{y} = -g - k\dot{s}\dot{y}.$$

From the first, $$\dot{x} = u\epsilon^{-ks},$$

and ∴ $$\dot{y} = p\dot{x} = up\epsilon^{-ks}, \text{ putting } p \text{ for } \frac{dy}{dx}.$$

Hence $$\ddot{y} = u^2 \frac{dp}{dx} \epsilon^{-2ks} - kup\dot{s}\epsilon^{-ks} = u^2 \frac{dp}{dx} \epsilon^{-2ks} - k\dot{s}\dot{y},$$

and the second equation becomes

$$\frac{dp}{dx} + \frac{g}{u^2} \epsilon^{2ks} = 0.$$

Multiplying by $2\dfrac{ds}{dx}$, which is equal to $2\sqrt{1+p^2}$, and integrating, we obtain

$$p\sqrt{1+p^2} + \log\left(p + \sqrt{1+p^2}\right) + \frac{g}{ku^2} \epsilon^{2ks} = C.$$

If then α be the initial inclination of the direction of motion, that is when $s = 0$, and if $p = \tan\theta$,

$$\tan\theta \sec\theta + \log(\tan\theta + \sec\theta) + \frac{g}{ku^2}(\epsilon^{2ks} - 1)$$
$$= \tan\alpha \sec\alpha + \log(\tan\alpha + \sec\alpha).$$

We observe that as s increases indefinitely, $\dfrac{dx}{dp}$ decreases and ultimately vanishes, shewing that x is ultimately constant, and therefore that the curve, on the positive side, tends to a vertical direction.

Again, if τ be the intercept of the tangent on the axis of y,

$$\tau = y - xp, \text{ and } \frac{d\tau}{dx} = -x\frac{dp}{dx} = -x\frac{g}{u^2}\epsilon^{2ks}.$$

Take x and s both negative, say $-x'$ and $-s'$; then as x' is certainly less than s',

$$\frac{x'}{\epsilon^{2ks'}} \text{ is less than } \frac{s'}{\epsilon^{2ks'}}$$

which vanishes when s' increases indefinitely.

Hence on the left-hand side, the value of $\frac{d\tau}{dx}$ approaches continually to zero, and τ is ultimately constant.

Again, if $s = -\infty$,

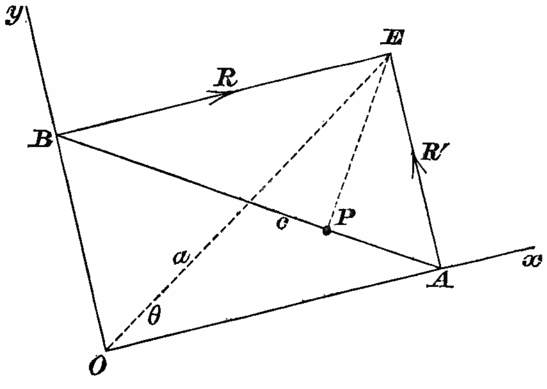
$$\tan\theta\sec\theta + \log(\tan\theta + \sec\theta) - C = 0,$$

shewing that θ has then a finite value; for, if we put $\theta = 0$, the left-hand number of the equation is negative, and if we put $\theta = \frac{\pi}{2}$, it is positive.

We infer then that the direction of the curve, on the negative side, is ultimately inclined to the vertical.

87. *Illustrations of the use of the equations of Arts.* (13) *and* (21).

(1) *Case of a weightless rod, of length 2a, carrying a particle at a distance c from its middle point, and having its ends moveable on two straight wires, at right angles to each other, and made to revolve uniformly in a horizontal plane.*



Taking the wires as axes of x and y, the equations of motion are

$$m(\ddot{x} - \omega^2 x - 2\omega\dot{y}) = R,$$
$$m(\ddot{y} - \omega^2 y + 2\omega\dot{x}) = R',$$

with the geometrical conditions,

$$x = (a + c)\cos\theta,\ y = (a - c)\sin\theta.$$

Now, since the system of acting forces is the equivalent of the system of effective forces, the resultant of R and R' must be in the line PE, and therefore

$$R(a + c)\sin\theta = R'(a - c)\cos\theta.$$

From these equations we obtain

$$(a^2 + c^2 - 2ac\cos 2\theta)\,\ddot{\theta} + 2ac\dot{\theta}^2\sin 2\theta + 2ac\omega^2\sin 2\theta = 0,$$

the integral of which is

$$(a^2 + c^2 - 2ac\cos 2\theta)\,\dot{\theta}^2 = C + 2ac\omega^2\cos 2\theta.$$

If the particle is at the centre of the rod, $\dot{\theta}$ is constant, so that, if initially the rod have no motion relative to the wires, its relative position will remain unchanged.

(2) *A smooth plane, carrying at a point C a centre of force the attraction to which varies as the distance, revolves uniformly about a fixed axis, perpendicular to itself; it is required to determine the motion of a particle on the plane.*

Taking the foot of the axis, O, for origin, and OC for the axis of x, the equations of motion are, if $OC = c$,

$$\ddot{x} - \omega^2 x - 2\omega\dot{y} = -\mu(x - c),$$
$$\ddot{y} - \omega^2 y + 2\omega\dot{x} = -\mu y.$$

The elimination of y leads to the symbolic equation

$$\{D^4 + 2(\omega^2 + \mu)D^2 + (\omega^2 - \mu)^2\}\,x = -\mu c(\omega^2 - \mu),$$

the solution of which is

$$x = -\frac{\mu c}{\omega^2 - \mu} + A\cos\{\overline{\omega + \sqrt{\mu}}\,t + \alpha\} + B\cos\{\overline{\omega - \sqrt{\mu}}\,t + \beta\};$$

and y is obtained in a similar form.

If it be required to determine the absolute path of the particle the equations of motion are

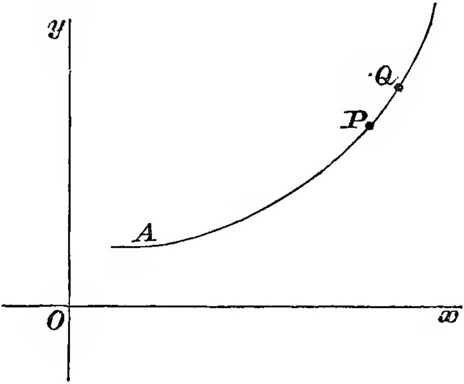
$$\ddot{x} = -\mu(x - c\cos\omega t),\ \ \ddot{y} = -\mu(y - c\sin\omega t),$$

the integrals of which are

$$x = \frac{\mu c}{\mu - \omega^2} \cos \omega t + A \cos (\sqrt{\mu} t + \alpha),$$

$$y = \frac{\mu c}{\mu - \omega^2} \sin \omega t + B \sin (\sqrt{\mu} t + \beta).$$

88. *Equations of motion of a free string in a plane under the action of given forces.*



If u and v be the velocities parallel to x and y, of a point P of the string, and T the tension at P, the component tensions are

$$T\frac{dx}{ds}, \text{ and } T\frac{dy}{ds}:$$

which are functions of the arc AP measured from a given point A of the string, and the equations of motion of the element δs are

$$m\delta s\, \dot{u} = \frac{d}{ds}\left(T\frac{dx}{ds}\right)\delta s + m\delta s \,.\, X,$$

$$m\delta s\, \dot{v} = \frac{d}{ds}\left(T\frac{dy}{ds}\right)\delta s + m\delta s \,.\, Y,$$

m being the mass of unit length of the chain, or

$$m\dot{u} = \frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX,$$

$$m\dot{v} = \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY.$$

The geometrical equation is given by the condition that the velocity of Q, in direction of the tangent at P, is ultimately the same as that of P, leading to

$$(u + \delta u)\frac{dx}{ds} + (v + \delta v)\frac{dy}{ds} = u\frac{dx}{ds} + v\frac{dy}{ds},$$

or
$$\frac{du}{ds}\frac{dx}{ds} + \frac{dv}{ds}\frac{dy}{ds} = 0.$$

These equations, combined with the boundary conditions, such for instance that the tension vanishes at a free end of the string, are sufficient theoretically for the determination of the motion of a string in a plane under the action of given forces.

The same equations apply to the problem of determining the initial tension at any point of a string, originally in equilibrium, when some of its constraints are removed.

It will be found however that this problem, and that of the next article, are more easily treated by making use of tangential and normal components.

89. *Motion of a string, or a fine chain, lying on a smooth horizontal plane, which has impulses applied at one or both ends.*

If T be the measure of the resulting impulsive tension at the point P, and u, v the velocities imparted to the point P,

$$m\delta s\,.\,u = \frac{d}{ds}\left(T\frac{dx}{ds}\right)\delta s, \qquad m\delta s\,.\,v = \frac{d}{ds}\left(T\frac{dy}{ds}\right)\delta s,$$

or
$$mu = T\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{dT}{ds}, \qquad mv = T\frac{d^2y}{ds^2} + \frac{dy}{ds}\frac{dT}{ds},$$

with the same geometrical condition as before, i.e.

$$\frac{du}{ds}\frac{dx}{ds}+\frac{dv}{ds}\frac{dy}{ds}=0.$$

EXAMPLES.

1. Find the law of force parallel to an asymptote under which a rectangular hyperbola can be described.

2. Prove that the vertical velocity of the centre of curvature of the path of a projectile is proportional to the time which has elapsed since the projectile was at the highest point of its orbit.

3. A particle moves in a plane under the action of forces to two fixed points in the plane, one attractive and the other repulsive, and each varying as the distance; if the absolute intensities be the same, find the path.

4. Find the path of a particle which is in motion under the action of a force perpendicular to a fixed line and inversely proportional to the square of the distance from the line.

5. A particle moves under the action of an attractive force perpendicular to a straight line and proportional to the distance from that line; prove that the path is the curve of sines.

6. Shew that the least inclination to the horizon at which a particle can be projected so as to strike at right angles any plane through the point of projection is $\cos^{-1}\frac{1}{3}$.

If the direction of projection be inclined at an angle θ to the plane, and if the projection on the plane of this direction be inclined at an angle ϕ to the line of greatest slope, shew that the range on the plane is

$$\frac{2v^2}{g}\frac{\sin\theta}{\cos^2\alpha}\left(\cos^2\theta\cos^2\alpha+\sin^2\theta\sin^2\alpha-\tfrac{1}{2}\sin 2\theta\sin 2\alpha\cos\phi\right)^{\frac{1}{2}},$$

where α is the inclination of the plane to the horizon.

7. Particles are projected from the same point in a vertical plane with velocities which vary as $(\sin\theta)^{-\frac{1}{2}}$, θ being the angle of projection; find the locus of the vertices of the parabolas described.

8. Particles fall down chords of a circle to the lowest point. Prove that the tangents to the circle at the upper extremities of the chords pass through the foci of the parabolas described after leaving the lowest point.

9. Particles slide down chords of a vertical circle to the lowest point; shew that the locus of the foci of the paths of the particles after leaving the chords is a cardioid.

10. An ellipse is held with its major axis vertical; find a point on the curve such that, if a perfectly elastic heavy particle slide down an inclined plane to it from the upper focus and be reflected by the curve, it will fall to the lower vertex; and shew that in an ellipse, whose eccentricity is 5, this point will be the extremity of the minor axis.

11. A bullet is fired in the direction towards a second equal bullet which is let fall at the same instant. Prove that the two bullets will meet, and that if they coalesce the latus-rectum of their joint path will be one quarter of the latus-rectum of the original path of the first bullet.

12. Chords are drawn, joining any point of a vertical circle with its highest and lowest points; prove that, if a heavy particle slide down the latter chord, the parabola, which it will describe after leaving the chord, will be touched by the former chord, and that the locus of the points of contact will be a circle.

13. Through a given point an inclined plane is drawn, perpendicular to a given vertical plane, and from that point a particle is projected in the vertical plane, with a given velocity, so as to strike the inclined plane at right angles; prove that the locus of the point on which it falls, for different positions of the inclined plane, is an ellipse.

14. Prove that the range of a projectile on an inclined plane is greatest for a given velocity of projection when the focus of the path is in the plane.

If t and t' be the two times of flight corresponding to any range short of the greatest, and α be the inclination of the plane, prove that

$$t^2 + t'^2 + 2tt' \sin \alpha$$

is independent of α.

15. Two smooth equal balls are placed in contact on a smooth table; a third equal ball strikes them simultaneously and remains at rest after the impact; shew that the coefficient of restitution is 2/3.

16. A person at a distance c from the vertical wall of a fives court discharges a perfectly elastic ball from his hand so that it shall strike first the floor, then the wall and finally return to his hand. If T be the whole time of motion and nT the time between the two rebounds, shew that the velocity and angle of projection measured downwards are given by

$$V \cos \alpha = 2c/T, \qquad V \sin \alpha = gnT.$$

17. On the moon there seems to be no atmosphere, and gravity is about one-sixth of that here on earth. What space of country would be commanded by the guns of a lunar fort, able to project shot at 1600 feet per second?

18. A particle is projected from a point in a smooth plane, inclined at an angle α to the horizon, in a vertical plane which cuts the inclined plane in a horizontal line, and at an angle θ to the horizon. Prove that after n rebounds the space travelled in the direction of the line of greatest slope on the inclined plane is

$$a \sin \alpha \tan \theta \, . \, \frac{e(1 - e^{n-1})}{1 - e},$$

where a is the horizontal space described, and e the coefficient of restitution.

19. A perfectly elastic particle is dropped from a point on a fixed vertical circular hoop; shew that after two rebounds it will rise vertically if

$$2 \sin 4\theta = \tan \theta,$$

where θ is the angular distance of the point from the highest point of the hoop.

20. The intersection of two planes, inclined to the horizon at the angles α, β, respectively, is horizontal: a perfectly elastic particle falls upon the former plane, then rebounds to the latter, and then rebounds vertically: prove that the ratio of the distance of the first point of impact from the intersection of the planes to the height through which the particle falls before striking the first plane is equal to

$$\frac{\cos(\alpha+\beta)}{\sin\beta \,.\, \sin 2\beta} \,.\, \{2\sin 2\alpha - \sin(2\alpha + 2\beta)\}.$$

21. A smooth inelastic ball slides from rest down a length (l) of a plane inclined 30° to the vertical, and impinges on a horizontal rail, parallel to the plane, and at a distance from it equal to half the radius of the ball. Neglecting the thickness of the rail, prove that the ball will afterwards strike the plane at a distance $3l$ from its point of contact when striking the rail.

22. A particle is projected from a point at the foot of one of two parallel vertical smooth walls so as after three reflections at the walls to return to the point of projection; prove that

$$e^3 + e^2 + e = 1,$$

and that the vertical heights of the three points of impact above the point of projection are as

$$e^2 : 1 - e^2 : 1.$$

23. A sphere is fixed upon a horizontal plane; find from what point in the plane a particle must be projected, with a velocity due to falling down a vertical space equal to the diameter of the sphere, so that the focus of its path may be in the centre; after reflection at the sphere shew that it will

strike the horizontal plane at a distance from the point of projection equal to the diameter of the sphere, if the elasticity be perfect.

24. Two equal perfectly elastic balls A and B are projected from the same point, in the same vertical plane and with equal horizontal velocities; prove that if when they first impinge A has not yet struck the ground, and B is moving horizontally, then the ratio of the cotangents of the angles which the directions of projection make with the vertical must $= \frac{2n^2+2n+1}{2n+1}$ where n is some positive integer, and also that when first after the impact one of them strikes the ground, the other is at a height above the ground

$$= \frac{2n(n+1)}{(2n+1)^2}\frac{v^2}{g},$$

where v is the vertical velocity of projection of B.

25. A ball is projected from a point in one of two smooth parallel vertical walls against the other in a plane perpendicular to both, and after being reflected at each wall impinges again on the second at a point in the same horizontal plane as it started from: shew that

$$be^2 = a(1+e+e^2),$$

where e is the coefficient of restitution, b the free range, and a the distance between the planes.

26. A man stands on the upper end of a long rough plank, of length a and mass M, which lies along a smooth straight groove on an inclined plane, having its upper end supported by a cord. The cord is cut, and, at the same instant, the man starts off, and runs with very short steps down the plank, at such a rate that the plank does not move; prove that the velocity of the man at the lower end of the plank is

$$\sqrt{2ga\cos\alpha\,\frac{m+M}{m}},$$

where m is the mass of the man, and α the inclination of the groove to the vertical.

If the man then jump horizontally so as not to set the plank in motion, he will alight on the groove at a distance

$$4a\cot^2\alpha\,.\,\frac{m+M}{m}$$

below the position of the lower end of the plank at the instant he alights.

Determine also how he must jump so that he may alight on the lower end of the plank.

27. A stone is projected upwards with velocity $\sqrt{2gc}$ from a point on the margin of a circular pond, radius c. If all directions of projection be equally probable, shew that the chance that the stone falls into the pond is

$$\tfrac{1}{2}-\frac{2(\sqrt{2}-1)}{\pi}.$$

28. A solid smooth cylinder, of radius r, lies on a smooth horizontal plane, to which it is fastened, and an inelastic sphere, of radius $2r$, moves along the plane in a direction at right angles to the axis of the cylinder; find the condition that it may pass over the cylinder.

If the sphere be elastic, and the coefficient of the elasticity be greater than $1/8$, prove that it cannot in any case pass over the cylinder; and if e be less than $1/8$, find the condition that the sphere may, after its first ascent, fall upon the top of the cylinder.

29. From a point A in one of two vertical lines a particle is projected with a velocity u at a given inclination to the horizon, and meets the other vertical line in B: it is then projected from B with a velocity v at the same inclination to the horizon and returns to A. Prove that the harmonic mean between u^2 and v^2 is constant.

30. A particle is projected horizontally along a plane, whose inclination (α) to the horizon is equal to the angle of friction (λ) between the particle and the plane; prove that it will ultimately move down the plane with a uniform velocity

equal to half the velocity of projection (V) and along a line, whose horizontal distance from the point of projection

$$= \tfrac{2}{3} \frac{V^2}{g \sin \alpha}.$$

If α be less than λ and $\tan \lambda = n \tan \alpha$, and if the particle receive a small horizontal impulse and when reduced to rest again, another, and so on continually, its path down the plane will not sensibly differ from a straight line inclined to the horizontal line in the plane at an angle $= \tan^{-1} \dfrac{4n^2 - 1}{8n(n^2 - 1)}$.

31. Find the path of a particle acted on by a repulsive force always perpendicular to a given straight line and proportional to the distance from it, the velocity at any point being that which would be acquired by moving from rest on the given line to that point.

32. If a particle be acted upon by a force always parallel to the axis of y and proportional to the square of the radius of curvature at the point, prove that it will describe the curve

$$\frac{y - b}{a} = \log \sec \frac{x}{a},$$

the particle moving parallel to the axis of x at the point (o, b).

33. The trochoid $x = a(\theta - e \sin \theta)$, $y = a(1 - e \cos \theta)$, is described under the action of a force parallel to the axis of x; shew that the force varies as $\dfrac{e - \cos \theta}{\sin^3 \theta}$.

34. If a particle be moving in a medium whose resistance varies as the velocity of the particle, shew that the equation of the trajectory referred to the vertical asymptote and a line parallel to the direction when the velocity is infinite as coordinate axes, is of the form

$$y = b \log \frac{x}{a}.$$

35. A body describes the curve whose equation is

$$\left.\overline{\frac{x}{a}}\right|^{n} + \left.\overline{\frac{y}{b}}\right|^{n} = 1,$$

under the action of a force to the centre. Shew that the central force is $\lambda r \, . \, (xy)^{n-2}$, and that when n is even, the periodic time

$$= \frac{2A}{(ab)^{\frac{n}{2}}}\sqrt{\frac{n-1}{\lambda}},$$

where A is equal to the area of the curve.

36. Two particles A and B, of masses $8m$ and m respectively, lie together at a point on a smooth horizontal plane, connected by a string which lies loose on the plane; B is projected at an elevation of 30° with velocity equal to g. If the string becomes tight the instant before B meets the plane again, and breaks when it has produced half the impulse it would have produced if it had not broken, and if the particle rebounds at an elevation of 30°, shew that the elasticity of B is equal to $5/9$.

37. A parabola, having its vertex at A and its axis coincident with AB the diameter of a semicircle, is described so as to cut the semicircle in P; prove that, if a body move in the semicircle under the action of a force perpendicular to AB, the time of moving from A to P varies as the difference between AB and the latus rectum. Prove also, that if a second body move from A to P in the parabola in the same time under the action of a force perpendicular to its axis, and the velocities in the two curves at P be equal, the latus rectum of the parabola is $\frac{1}{3} AB$.

38. A parabola is described under an acceleration f to the focus, and another a parallel to the axis; shew that

$$\frac{d}{dr} \cdot (f + a) + \frac{2f}{r} = 0.$$

39. Two particles are projected in parallel directions from two points in a straight line passing through a centre of

force, the attraction to which varies as the distance, with velocities proportional to their distances from the centre. Prove that all tangents, to the path of the inner, cut off, from that of the outer, arcs described in equal times.

40. *OA* is a smooth tube; *OB* a light rod perpendicular to it; *B*, a fixed point in *OB*, a centre of force attracting with force μr a particle P in the tube *OA*. The system being made to revolve with uniform angular velocity ω on a horizontal plane about O, determine the motion of P; and shew that, if $\mu > \omega^2$, P will oscillate with period $2\pi/\sqrt{\mu - \omega^2}$.

41. A rod revolves about its middle point with uniform angular velocity ϖ and has at its extremities two centres of force varying as the distance one attractive and one repulsive of the same absolute intensity; supposing a particle placed in the plane of rotation in a line perpendicular to the rod through its centre, shew that its path will be cycloidal, the time from one cusp to another being $2\pi/\varpi$.

42. A smooth horizontal disc rotates with angular velocity $\sqrt{\mu}$ about a vertical axis at which is placed a particle attracted to a certain point of the disc by a force whose acceleration is $\mu \times$ distance; prove that the path of the particle on the disc will be a cycloid.

43. A particle moves under the action of two constant forces in the ratio of nine to one, whose directions rotate in opposite directions with uniform angular velocities in the ratio of three to one: prove that, under certain initial conditions, the path of the particle will be a closed curve, of the same form as that represented by the equation, $r = a \cos 2\theta$.

44. The two ends of a smooth weightless rod are moveable on two fixed straight wires intersecting each other at right angles. A particle can move on the rod and is attracted to the point of intersection of the wires by a force varying as the distance. Prove that if the particle have initially no motion the angular velocity of the rod is given by an equation of the form

$$\omega^2 = n^2 \{1 - \sin^2 2\alpha \operatorname{cosec}^2 2\theta\}.$$

CHAPTER VII.

RADIAL AND TRANSVERSAL ACCELERATIONS.

90. HAVING discussed, in the previous chapter, the use of the components of acceleration parallel to two coordinate axes, we now take into consideration the expressions for radial and transversal components, leading to the equations of motion,

$$\ddot{r} - r\dot{\theta}^2 = P, \ \ r\ddot{\theta} + 2\dot{r}\dot{\theta} = Q,$$

mP and mQ being the radial and transversal forces acting on a particle of mass m.

For our first illustration we take the following case.

Motion of a particle in a smooth straight tube which revolves uniformly, in a horizontal plane, about a fixed point in the axis of the tube.

In this case the only force acting on the particle is the pressure R of the tube, and, if ω be the angular velocity, the equations are

$$\ddot{r} - \omega^2 r = 0, \ \ 2m\dot{r}\omega = R.$$

If the particle start from the distance a with no initial velocity along the tube, we obtain from the first equation,

$$r = \frac{a}{2}(\epsilon^{\omega t} + \epsilon^{-\omega t}) = a \cosh . \omega t,$$

and from the second,

$$R = 2ma\omega^2 \sinh . \omega t.$$

We can also obtain the velocity and pressure in terms of r, for the first equation gives

$$\dot{r}^2 = \omega^2 (r^2 - a^2),$$

and therefore $$R = 2m\omega^2 \sqrt{r^2 - a^2}.$$

If b be the length of the tube, the direction in which the particle flies out is inclined to the tube at an angle θ such that

$$\tan\theta = \frac{b\omega}{\omega\sqrt{b^2 - a^2}} = \frac{b}{\sqrt{b^2 - a^2}}.$$

If the tube revolve in a vertical plane the equations are

$$\ddot{r} - \omega^2 r = -g \sin\omega t,\ \ 2m\dot{r}\omega = R - mg\cos\omega t.$$

From these equations,

$$r = \frac{g}{2\omega^2}\sin\omega t + A\epsilon^{\omega t} + B\epsilon^{-\omega t} \qquad \text{(Chapter II.)},$$

and $$R = 2mg\cos\omega t + 2m\omega^2 (A\epsilon^{\omega t} - B\epsilon^{-\omega t}),$$

the constants being determined by initial conditions.

91. *Motion of a particle in a straight tube which revolves uniformly in a horizontal plane about a fixed point at a distance c from its axis.*

If OA (c) be the perpendicular from the fixed point on the axis of the tube, and, P being the position of the particle, if $AP = r$, the acceleration of P relative to $A = \ddot{r} - \omega^2 r$, in the direction AP, and $2\dot{r}\omega$ perpendicular to AP; and the acceleration, $\omega^2 c$, of the point A is wholly in the direction AO.

Hence the equations of motion are

$$\ddot{r} - \omega^2 r = 0,\ \ m(2\dot{r}\omega + \omega^2 c) = R,$$

and the solution is similar to that of the preceding case.

Motion of two particles, m and m′, connected together by an inelastic string, in a straight tube revolving uniformly in a horizontal plane about one end.

If r be the distance of m from the origin, l the length of string, and T its tension, the equations are

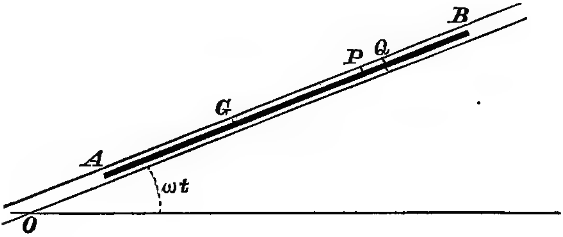
$$m(\ddot{r} - \omega^2 r) = T, \quad m'\{\ddot{r} - \omega^2(r + l)\} = -T,$$

from which we obtain

$$(m + m')\ddot{r} - \omega^2(m + m')r - m'\omega^2 l = 0,$$

and therefore $$r + \frac{m'l}{m + m'} = A\epsilon^{\omega t} + B\epsilon^{-\omega t}.$$

92. *Motion of a piece of uniform chain in a straight tube revolving in the same manner.*



Let r be the distance from O of G, the centre of gravity of the chain, $GP = \rho$ the distance from G of one end of an element PQ, $(\delta\rho)$ of the chain, m the mass per unit of length, T the tension at P, $T + \delta T$ at Q.

The equations of motion of the element are, since ρ is independent of the time and therefore $\frac{d^2}{dt^2}(OP)$ equal to $\ddot{r}$,

$$m\delta\rho\{\ddot{r} - \omega^2(r + \rho)\} = \delta T, \quad 2m\delta\rho\dot{r}\omega = R\delta\rho,$$

R being the rate of pressure at P per unit of length.

Integrating the first of these equations, or, in other words, taking the sum of the equations of motion of all the elements, we obtain

$$m\left\{(\ddot{r} - \omega^2 r)\rho - \omega^2\frac{\rho^2}{2}\right\} = T + C.$$

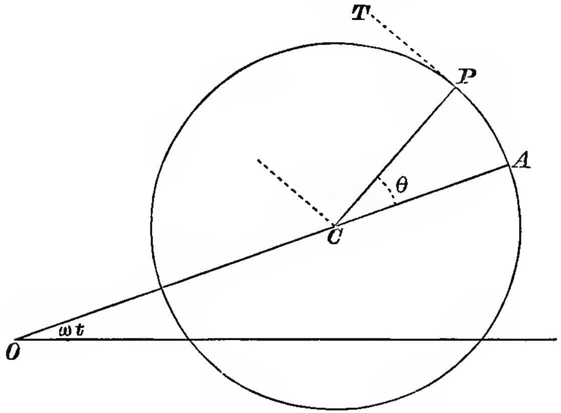
Observing that $T=0$ when $\rho=-l$ and when $\rho=l$, $2l$ being the length of the chain, there results

$$m(\ddot{r}-\omega^2 r)\,2l=0, \text{ or } \ddot{r}-\omega^2 r=0,$$

shewing that the motion of the centre of gravity is the same as that of a single particle.

Taking account of this result, $T=\frac{1}{2}m\omega^2(l^2-\rho^2)$.

93. *Motion of a bead on a smooth circular wire revolving uniformly in its own plane about a fixed point.*



If O be the fixed point, and θ the angle PCA, the accelerations of P in the directions PT and PC, obtained by compounding the accelerations of P relative to C with that of C relative to O, are respectively,

$$a\frac{d^2}{dt^2}(\theta+\omega t)+\omega^2 c\sin\theta, \text{ and } a(\dot{\theta}+\omega)^2+\omega^2 c\cos\theta,$$

taking $OC=c$, and $CP=a$.

If the plane be horizontal, and R be the pressure of the wire on the bead, the equations of motion are

$$a\ddot{\theta}+\omega^2 c\sin\theta=0, \text{ and } m\{a(\dot{\theta}+\omega)^2+\omega^2 c\cos\theta\}=R.$$

If the bead be originally attached to the wire at the angular distance α from the line CA, and be set free, the first of these equations leads to

$$a\dot{\theta}^2 = 2\omega^2 c\,(\cos\theta - \cos\alpha),$$

shewing that the bead oscillates through the angle 2α, and the second determines the pressure.

CENTRAL FORCES.

94. If a particle move under the action of an attractive force mP to a fixed centre, P being a function of r, the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = -P,\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

From the second equation we at once obtain

$$r^2\dot{\theta} = h,$$

h being a constant.

Hence $$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{du}{d\theta},\ \text{if}\ u = \frac{1}{r}:$$

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2},$$

and the first equation becomes

$$h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = P,$$

or $$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}.$$

This equation, if the law of force be given, determines the path, and, if the path be given, determines the law of force.

95. If A be the area swept over by the radius vector,

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta} = \tfrac{1}{2}h,$$

and therefore $A = \frac{1}{2}ht$, shewing that the area is proportional to the time, and that $\frac{1}{2}h$ is the area swept over in the unit of time.

If δs be an element of the arc of the curve, and p the perpendicular from the centre of force upon the tangent,

$$\delta A = \tfrac{1}{2}p\delta s,$$

and therefore $$h = 2\dot{A} = p\dot{s} = pv,$$

shewing that the velocity is inversely proportional to the distance of the centre from the tangent to the path.

96. We have shewn, in Art. (17), that the normal acceleration, in any curvilinear path, is equal to v^2/ρ, where ρ is the radius of curvature.

If mF be the resultant of the forces acting on a particle m, and ϕ the inclination to the normal of the line of action of this resultant, it follows that

$$m\frac{v^2}{\rho} = mF\cos\phi,$$

and therefore, if q be the chord of curvature in direction of the force,

$$v^2 = \tfrac{1}{2}Fq.$$

That is, *the velocity of a particle at any point of an orbit is that which the particle would acquire if it were to move from rest under the action of a constant force, equal to the force at the point, through a space equal to one-fourth of the chord of curvature in direction of the force.*

97. Since $$v = \frac{h}{p}, \text{ and } \cos\phi = \frac{p}{r},$$

we have $$\frac{h^2}{p^2} = P\rho\frac{p}{r} = Pp\frac{dr}{dp},$$

and therefore $$P = \frac{h^2}{p^3}\frac{dp}{dr},$$

an expression which is frequently useful in determining the path for a given law of force, or the law of force for a given path.

We can also obtain the same expression for P by employing the expression for the tangential acceleration.

Thus $$-P\sin\phi = v\frac{dv}{ds} = v\frac{dv}{dr}\sin\phi;$$

therefore $$P = -\frac{h}{p}\frac{d}{dr}\left(\frac{h}{p}\right) = \frac{h^2}{p^3}\frac{dp}{dr}.$$

The two expressions for P are deducible, each from the other, by help of the equation,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

98. Another expression for the velocity is found by utilizing the expression for the tangential acceleration; we thus obtain

$$v\frac{dv}{ds} = -P\frac{dr}{ds},$$

and therefore $$v^2 = v'^2 - 2\int P dr,$$

shewing that v is a function of the distance.

Further, since

$$v^2 = \frac{h^2}{p^2} = h^2\left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\},$$

it follows that $r\dfrac{d\theta}{dr}$ is a function of the distance, and therefore that at all points which are at the same distance from the centre of force the angle between the radius vector and the tangent is the same.

We now proceed to apply these formulæ to some particular cases.

99. *To find the law of force to the focus under the action of which a conic section can be described.*

Taking as the equation of the conic

$$cu = 1 + e\cos\theta,$$

$$c\frac{d^2u}{d\theta^2} = -e\cos\theta, \text{ and } \frac{d^2u}{d\theta^2} + u = \frac{1}{c},$$

so that $$P = \frac{h^2u^2}{c} = \frac{h^2}{c} \cdot \frac{1}{r^2},$$

and therefore varies inversely as the square of the distance.

If μ be the absolute force, that is the force at the unit of distance on a particle of unit mass, $\mu = \dfrac{h^2}{c} = \dfrac{2h^2}{L}$, if L be the latus rectum.

To find the law of force to the centre under which a central conic can be described.

Employing the equation, $P = \dfrac{h^2}{p^3}\dfrac{dp}{dr}$, we know that in this case,

$$p^2(a^2 + b^2 - r^2) = a^2b^2,$$

and therefore $-r\dfrac{dr}{dp} = -\dfrac{a^2b^2}{p^3}$, so that $P = \dfrac{h^2}{a^2b^2}r$,

and therefore varies as the distance.

If μ be the absolute force, $\mu = \dfrac{h^2}{a^2b^2}$.

100. *To find the law of force to the pole under which an equiangular spiral can be described.*

From the definition of the curve, $p = r\sin\alpha$,

and therefore $$P = \frac{h^2}{\sin^2\alpha} \cdot \frac{1}{r^3} = \frac{\mu}{r^3},$$

and the velocity $$= \frac{h}{p} = \frac{\sqrt{\mu}}{r}.$$

To find the law of force when a particle describes a circle under the action of a force to a point in the circumference.

In this case $\dfrac{p}{r} = \dfrac{r}{2a}$, or $r^2 = 2ap$,

and therefore $P = \dfrac{h^2}{p^3} \cdot \dfrac{r}{a} = \dfrac{8h^2a^2}{r^5} = \dfrac{\mu}{r^5}$

and the velocity $= \dfrac{2ah}{r^2} = \dfrac{1}{r^2}\sqrt{\dfrac{\mu}{2}}$.

101. *Motion of a particle under the action of a force to a fixed point varying inversely as the square of the distance.*

In this case, $P = \mu u^2$, and we obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

the integral of which is

$$u - \frac{\mu}{h^2} = A\cos(\theta - \gamma),$$

or

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \gamma)\},$$

which is the equation of a conic section, of which e is the eccentricity, and $\dfrac{h^2}{\mu}$ is the semi-latus rectum.

To find the constants, let c be the initial distance, v the velocity of projection, and β the inclination to c of the initial direction of motion; then

$$h = vc\sin\beta, \text{ and, since } \frac{du}{d\theta} = -\frac{\mu}{h^2}e\sin(\theta - \gamma),$$

$$\frac{1}{c} = \frac{\mu}{h^2}(1 + e\cos\gamma) \text{ and } \frac{1}{c}\cot\beta = -\frac{\mu}{h^2}e\sin\gamma,$$

or $v^2c\sin^2\beta = \mu(1 + e\cos\gamma)$, $v^2c\sin\beta\cos\beta = -\mu e\sin\gamma$;

whence $$(v^2c\sin^2\beta-\mu)^2+v^4c^2\sin^2\beta\cos^2\beta=\mu^2e^2,$$

or $$e^2=1+\frac{v^4c^2\sin^2\beta}{\mu^2}-\frac{2v^2c\sin^2\beta}{\mu};$$

and $$\tan\gamma=\frac{v^2c\sin\beta\cos\beta}{\mu-v^2c\sin^2\beta}.$$

It follows that the conic is an ellipse, parabola, or hyperbola, according as v^2 is less than, equal to, or greater than $\frac{2\mu}{c}$.

If $2a$ and $2b$ be the axes when the curve is an ellipse,

$$\frac{h^2}{\mu}=a(1-e^2);$$

and therefore

$$\frac{v^2c^2\sin^2\beta}{\mu}=a\left(\frac{2v^2c\sin^2\beta}{\mu}-\frac{v^4c^2\sin^2\beta}{\mu^2}\right),$$

or $$v^2=\frac{2\mu}{c}-\frac{\mu}{a}.$$

Since any point may be regarded as the point of projection, the velocity at the distance r is given by the equation

$$v^2=\frac{2\mu}{r}-\frac{\mu}{a}.$$

In the same manner, if the conic be a hyperbola, we find that

$$v^2=\frac{2\mu}{r}+\frac{\mu}{a}.$$

We can also solve this question by the use of the equation

$$\frac{h^2}{p^3}\frac{dp}{dr}=P=\frac{\mu}{r^2},$$

leading to $$\frac{h^2}{p^2}=\frac{2\mu}{r}+C,$$

which is an ellipse, parabola, or hyperbola, according as C is negative, zero, or positive.

If $C=0$, the velocity $=\frac{h}{p}=\sqrt{\frac{2\mu}{r}}$.

If the curve be an ellipse or hyperbola, the axes of which are $2a$ and $2b$, we find by comparison with the equation

$$\frac{b^2}{p^2}=\frac{2a\mp r}{r},$$

that $\mu=\frac{h^2a}{b^2}=\frac{2h^2}{L}$, if L be the latus rectum, and that

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}\mp\frac{\mu}{a}.$$

102. The case in which the force varies as the distance has already been dealt with in Art. (81), but we can also usefully employ the equation

$$\frac{h^2}{p^3}\frac{dp}{dr}=P=\mu r,$$

leading to

$$\frac{h^2}{p^2}=C-\mu r^2,$$

which is the equation of an ellipse.

Comparing with the equation

$$p^2(a^2+b^2-r^2)=a^2b^2,$$

we find that $C=\mu(a^2+b^2)$, and $h^2=\mu a^2b^2$, and that

$$v^2=\frac{h^2}{p^2}=\mu(a^2+b^2-r^2)=\mu\,.\,CD^2,$$

if CD be conjugate to r.

103. *Motion of a particle under the action of a repulsive force from a fixed point varying inversely as the square of the distance from that point.*

The equation of motion is

$$\frac{d^2u}{d\theta^2}+u=-\frac{\mu}{h^2};$$

therefore
$$u + \frac{\mu}{h^2} = A\cos(\theta - \gamma),$$

or
$$\frac{h^2}{\mu} \cdot \frac{1}{r} = -1 + e\cos(\theta - \gamma).$$

Introducing the initial conditions, we find that

$$\frac{v^2 c \sin^2\beta}{\mu} + 1 = e\cos\gamma \text{ and } \frac{v^2 c \sin\beta\cos\beta}{\mu} = -e\sin\gamma;$$

therefore
$$e^2 = 1 + \frac{2v^2 c \sin^2\beta}{\mu} + \frac{v^4 c^2 \sin^2\beta}{\mu^2},$$

shewing that the conic is a hyperbola.

As in the previous case,

$$\frac{h^2}{\mu} = a(e^2 - 1)$$

leads to
$$\frac{v^2 c^2 \sin^2\beta}{\mu} = a\left(\frac{2v^2 c \sin^2\beta}{\mu} + \frac{v^4 c^2 \sin^2\beta}{\mu^2}\right),$$

and therefore
$$v^2 = \frac{\mu}{a} - \frac{2\mu}{c}.$$

Hence, any point being a point of projection, we have at the distance r

$$v^2 = \frac{\mu}{a} - \frac{2\mu}{r}.$$

104. *Path of a particle projected at the distance a with the velocity $\sqrt{\mu} \div a$ in the direction at right angles to the initial distance, and subject to the action of a central force which on unit mass is equal to*

$$\mu\{2(a^2 + b^2)u^5 - 3a^2 b^2 u^7\}.$$

In this case $h = \frac{\sqrt{\mu}}{a} . a = \sqrt{\mu}$, and therefore

$$\frac{d^2u}{d\theta^2} + u = 2(a^2 + b^2)u^3 - 3a^2 b^2 u^5.$$

Multiplying by $2\dfrac{du}{d\theta}$, integrating and observing that when $u = \dfrac{1}{a}$, $\dfrac{du}{d\theta} = 0$, we find

$$\left(\frac{du}{d\theta}\right)^2 = u^2\,(a^2u^2 - 1)\,(1 - b^2u^2),$$

restoring r this becomes

$$\frac{d\theta}{dr} = \frac{r}{\sqrt{(a^2 - r^2)\,(r^2 - b^2)}}.$$

Putting $z^2 = r^2 - b^2$, and integrating, we obtain

$$r^2 - b^2 = (a^2 - b^2)\cos^2(\theta - \gamma).$$

But $r = a$ when $\theta = 0$, and therefore $\gamma = 0$, and

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

is the orbit, which is the pedal of an ellipse with regard to its centre.

VELOCITY IN AN ORBIT.

105. If the orbit be central, the velocity is given by the equation

$$v = \frac{h}{p}.$$

In general, whether the orbit be central or not, the velocity is given by the equation,

$$v^2 = \tfrac{1}{2}Fq,$$

where q is the chord of curvature in the direction of the resultant force.

For instance if the orbit be an ellipse, the force being directed to the centre C, and if v be the velocity at P,

$$v^2 = \tfrac{1}{2}\mu CP \,.\, \frac{2\,CD^2}{CP} = \mu CD^2.$$

For an ellipse, when the force is to the focus,

$$v^2 = \tfrac{1}{2}\frac{\mu}{SP^2}\cdot\frac{2CD^2}{AC} = \frac{2\mu}{SP} - \frac{\mu}{AC}.$$

For a parabola, force to the focus,

$$v^2 = \tfrac{1}{2}\,\frac{\mu}{SP^2}\cdot 4SP = \frac{2\mu}{SP}.$$

For a hyperbola, force to the focus,

$$v^2 = \tfrac{1}{2}\frac{\mu}{SP^2}\cdot\frac{2CD^2}{AC} = \frac{2\mu}{SP} + \frac{\mu}{AC}.$$

For a hyperbola when the force is repulsive from the ocus,

$$v^2 = \tfrac{1}{2}\cdot\frac{\mu}{Sp^2}\cdot\frac{2CD^2}{AC} = \frac{\mu}{AC} - \frac{2\mu}{SP}.$$

For an equiangular spiral, force to the pole,

$$v^2 = \tfrac{1}{2}\,\frac{\mu}{r^3}\,2r = \frac{\mu}{r^2}.$$

For the orbit,

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta, \qquad \text{Art. (104)},$$

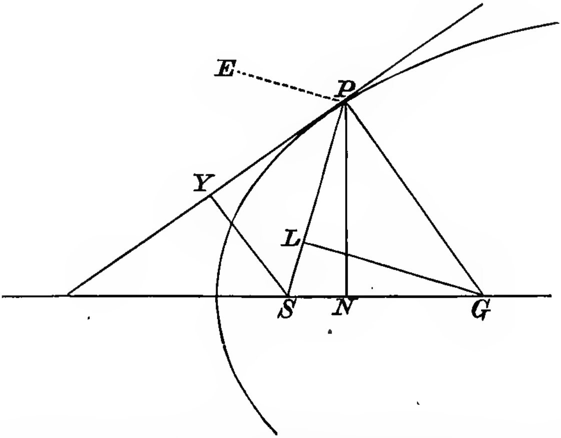
will be found that

$$p = \frac{r^3}{\sqrt{(a^2+b^2)\,r^2 - a^2b^2}}, \text{ and } \therefore\ v = \frac{\sqrt{\mu}}{r^3}\sqrt{(a^2+b^2)\,r^2 - a^2b^2}.$$

106. *If a particle move in a conic section under the ction of a force to the focus, the velocity at any point can be ecomposed into two constant velocities, one perpendicular the radius vector, and the other perpendicular to the axis of e conic.*

The sides of the triangle SPG are perpendicular to the rection of the actual velocity, and to the directions in uestion. Hence the velocity, and its components are in e ratio of the sides of the triangle. GL being drawn perpendicular to SP, the component perpendicular to SP

$$= \frac{h}{SY} \cdot \frac{SP}{PG} = \frac{h}{PL} = \frac{h}{\frac{1}{2}L} = \frac{\mu}{h}, \text{ Art. (101).}$$



and, since $SG = e \,.\, SP$, the component perpendicular to the axis $= \dfrac{e\mu}{h}$.

It will be noticed that the latter component is in the direction PN, or NP, according as the body is moving towards, or away from, the vertex.

TIME IN AN ORBIT.

107. The time of passing from one point to another of a central orbit is in all cases determined by the equation, $r^2 \dot{\theta} = h$, that is by the fact that the area swept over by the radius vector is proportional to the time.

Time of traversing an arc of a parabolic path, when the force is to the focus.

The parabola being

$$r = \frac{2a}{1+\cos\theta} = a\sec^2\frac{\theta}{2},$$

we have $\dfrac{dt}{d\theta} = \dfrac{r^2}{\sqrt{2\mu a}} = \dfrac{a^{\frac{3}{2}}}{\sqrt{2\mu}}\sec^4\dfrac{\theta}{2} = \dfrac{a^{\frac{3}{2}}}{\sqrt{2\mu}}\left(1+\tan^2\dfrac{\theta}{2}\right)\sec^2\dfrac{\theta}{2},$

and therefore $\qquad t = a^{\frac{3}{2}}\sqrt{\dfrac{2}{\mu}}\left(\tan\dfrac{\theta}{2} + \dfrac{1}{3}\tan^3\dfrac{\theta}{2}\right),$

measuring from the vertex.

108. *Time in an ellipse when the force is to the centre.*

The equation of the ellipse being

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2},$$

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{r^2}{ab\sqrt{\mu}} = \frac{1}{\sqrt{\mu}}\,\frac{ab}{a^2\sin^2\theta + b^2\cos^2\theta},$$

$$= \frac{ab}{\sqrt{\mu}}\,\frac{\sec^2\theta}{b^2 + a^2\tan^2\theta},$$

and therefore the time from the end of the transverse axis

$$= \frac{1}{\sqrt{\mu}}\tan^{-1}\frac{a\tan\theta}{b}.$$

The Periodic time is equal to the area of the curve divided by $\frac{1}{2}h$, and

$$= \frac{\pi ab}{\frac{1}{2}ab\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}},$$

It will be noticed that this is independent of the size of the orbit.

109. *Time in an ellipse when the force is to the focus.*

Taking $\frac{c}{r} = 1 + e\cos\theta$ as the equation of the curve, we have

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{r^2}{\sqrt{\mu c}} = \frac{c^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \frac{1}{(1+e\cos\theta)^2}.$$

Now $e\int\frac{d\theta}{(1+e\cos\theta)^2} = \int\frac{\cos\theta + e}{(1+e\cos\theta)^2}\,d\theta - \int\frac{\cos\theta\,d\theta}{(1+e\cos\theta)^2}$

$$= \frac{\sin\theta}{1+e\cos\theta} - \frac{1}{e}\int\left\{\frac{1}{1+e\cos\theta} - \frac{1}{(1+e\cos\theta)^2}\right\}d\theta\,;$$

therefore

$$(1-e^2)\int\frac{d\theta}{(1+e\cos\theta)^2} = -\frac{e\sin\theta}{1+e\cos\theta} + \int\frac{d\theta}{1+e\cos\theta}$$

$$= -\frac{e\sin\theta}{1+e\cos\theta} + \frac{2}{\sqrt{1-e^2}}\tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right\},$$

and the time from the vertex is therefore, since $c = a\sqrt{1-e^2}$,

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right\}.$$

The total area being πab, the periodic time

$$= \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu a(1-e^2)}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

This of course can be obtained from the preceding expression by taking $\theta = \pi$, and doubling the result.

110. *Time in a hyperbola, when the force is to the focus.*

The process is exactly the same, only that, e being greater than unity, the result of the integration appears in the form

$$(e^2-1)\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{e\sin\theta}{1+e\cos\theta}-\frac{1}{\sqrt{e^2-1}}\log\left\{\frac{1+\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}}{1-\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}}\right\}.$$

111. *Time in the orbit,* $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

In this case $h=\sqrt{\mu}$, $\sqrt{\mu}t=\int r^2 d\theta$;

therefore $\sqrt{\mu}t=\frac{1}{2}(a^2+b^2)\theta+\frac{1}{4}(a^2-b^2)\sin 2\theta,$

and the periodic time $=\dfrac{\pi(a^2+b^2)}{\sqrt{\mu}}$.

APSES AND APSIDAL DISTANCES.

112. *An apse in a central orbit is a point at which the tangent is perpendicular to the radius vector, and the length of the radius vector at the point is the apsidal distance.*

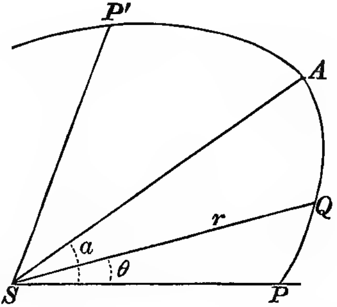
We have shewn that, if the central force be a function of the distance, the velocity at any point and the inclination of the tangent to the radius vector are also functions of the distance, and from these facts it follows that if the motion at any point be reversed in direction, the particle will retrace its path in the opposite direction.

For, supposing the particle projected from P with a given velocity v to arrive at the apse A with a velocity u, the value of $r\dfrac{d\theta}{dr}$ is a function of v and r, and therefore if

$$\frac{d\theta}{dr}=f(r),\quad \theta=\int_{SP}^{SQ}f(r)\,dr,$$

and if $ASP=\alpha$, $\alpha=\int_{SP}^{SA}f(r)\,dr.$

Now, suppose the motion reversed at A; then the values of θ and a are the same as before, and the orbit is therefore retraced.



We hence see that *any apsidal line divides the orbit symmetrically.*

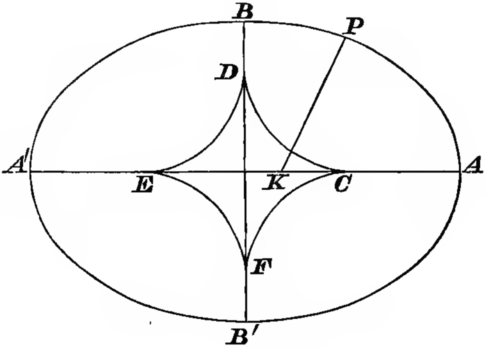
For on arriving at A the particle is under the same circumstances with regard to the direction AP' as it was when reversed with regard to the direction AP.

Hence it is obvious that, at an apse, the radius vector has a maximum or minimum value, and further that, in a central orbit in which the force is a unique function of the distance, *there can only be two different apsidal distances, although there may be any number of apses.*

113. It follows that we cannot ensure the complete description of an orbit by placing a centre of force at any assigned point. Take for instance the case of an ellipse, and trace its evolute.

The centre of force may be at the centre O, or at any point of each of the four limited lines AC, DB, EA', FB'.

If a centre of force be placed anywhere else, as at K, the normal KP does not divide the orbit symmetrically, and although a particle, projected from A, may describe the arc AP, it will not proceed in the ellipse, but will describe the arc PA turned over to the other side of KP.



The same remarks apply to an orbit of any form.

114. If the law of attraction be inversely as the nth power of the distance, that is, if $P = \mu u^n$, the equation of motion is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u^{n-2},$$

leading to

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu}{h^2} \frac{u^{n-1}}{n-1} + C.$$

Hence the apsidal distances are given by the equation

$$u^2 = \frac{2\mu}{h^2} \frac{u^{n-1}}{n-1} + C,$$

which cannot have more than two positive roots, a result in accordance with that of Art. (112).

If however the moving particle be so projected that

$$h^2 = \frac{\mu}{c^{n-3}}, \text{ and } C = \frac{n-3}{(n-1)c^2},$$

we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2c^{n-3}u^{n-1}}{n-1} + \frac{n-3}{(n-1)c^2}.$$

and the equation for the apsidal distances takes the form

$$(n-1)c^2u^2 = 2(cu)^{n-1} + n - 3.$$

It is easily seen by taking the derived function that this equation has two equal roots, that is, that there is really only one apsidal distance, of length c.

Taking $n = 5$ for example, we find that

$$\left(\frac{du}{d\theta}\right)^2 = \frac{(c^2u^2-1)^2}{2c^2},$$

and therefore $$\left(\frac{d\theta}{dr}\right)^2 = \frac{2c^2}{(c^2-r^2)^2}.$$

Integrating we find that the equation of the orbit is

$$\frac{r}{c} = \frac{2^{\sqrt{2}(\theta-\alpha)} - 1}{2^{\sqrt{2}(\theta-\alpha)} + 1},$$

a curve which has an asymptotic circle of radius c.

Hence it appears that, when projected in the manner described, the particle will make an infinite number of revolutions before arriving at the apsidal distance of its path.

If for another example we take $n = 2$, the equation of the path is

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\frac{u}{c} - \frac{1}{c^2}$$

or $$\left(\frac{du}{d\theta}\right)^2 + \left(u - \frac{1}{c}\right)^2 = 0,$$

shewing that $$\frac{du}{d\theta} = 0, \text{ and } u = \frac{1}{c},$$

or that the only possible path, under the conditions stated, is a circle having the centre of attraction for its centre.

In fact it will be found impossible to satisfy the conditions for any angle of projection which is not a right angle.

115. The peculiarity of the preceding case is that the velocity of the apse is that which is suitable for circular motion round the centre of attraction as a centre.

$$\text{For } v^2 = \frac{h^2}{p^2} = h^2 \left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\}$$

$$= \frac{\mu}{(n-1)\,c^{n-1}} \{2\,(cu)^{n-1} + n - 3\}$$

$$= \frac{\mu}{c^{n-1}} \text{ when } r = c.$$

116. *A particle describes a nearly circular orbit about a centre of force; it is required to find approximately the equation of the path and the apsidal angle.*

If $\phi(r)$ be the force, and if the particle be projected at the distance c, perpendicularly to the distance with the velocity $\sqrt{c\phi(c)}$, it will describe a circle.

We shall suppose the particle projected perpendicularly with the velocity $\sqrt{c\phi(c)}$ at the distance $c+\gamma$, γ being a very small quantity.

We have then $h^2 = (c+\gamma)^2\, c\phi(c)$, and if we suppose that $r = c + x$, where x is very small,

$$u = \frac{1}{r} = \frac{1}{c} - \frac{x}{c^2},$$

and

$$\frac{\phi(r)}{h^2u^2} = \frac{\phi(c+x)\,(c+x)^2\,(c+\gamma)^{-2}}{c\phi(c)}$$

$$= \frac{1}{c}\left(1 + \frac{2x}{c} + \frac{x\phi'(c)}{\phi(c)} - \frac{2\gamma}{c}\right),$$

if we neglect the squares of the small quantities.

The differential equation then becomes

$$\frac{d^2x}{d\theta^2}+x\left\{3+\frac{c\phi'(c)}{\phi(c)}\right\}=2\gamma,$$

and the approximate equation of the path is

$$r-c=\frac{2\gamma}{3+\dfrac{c\phi'(c)}{\phi(c)}}+A\cos\left\{\sqrt{3+\frac{c\phi'(c)}{\phi(c)}}\,\theta+\alpha\right\},$$

at the apses, $\frac{du}{d\theta}=0$, and therefore the apsidal angle is

$$\frac{\pi}{\sqrt{3+\dfrac{c\phi'(c)}{\phi(c)}}}.$$

If $\phi(r)=\mu r$ the apsidal angle is $\frac{\pi}{2}$, and if $\phi(r)=\frac{\mu}{r^2}$ it is equal to π.

It will be seen that if the force vary as the nth power of the distance n must not be less than -3.

In other words if n is greater than -3, a circular orbit possesses the characteristic of stability.

If $n=3$, the apsidal angle is infinite. In this case if the particle be describing a circle about the centre of attraction as centre, any slight divergence of path, without change of velocity, will cause the particle to describe an equiangular spiral.

117. *Case in which $\phi(r)=\mu r^n$, where n is a positive integer, the particle being projected from an apse at the distance c with the velocity $\sqrt{\mu c^{n+1}}$.*

In this case the equation of motion is

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{c^{n+3}\,u^{n+2}},$$

leading to

$$\left(\frac{du}{d\theta}\right)^2+\frac{(n+1)\,(cu)^{n+3}+2-(n+3)\,(cu)^{n+1}}{(n+1)\,c^{n+3}\,u^{n+1}}=0.$$

Now if we take

$$f(x) = (n+1)\,x^{n+3} + 2 - (n+3)\,x^{n+1},$$

we find that the equation, $f(x) = 0$, has two roots equal to unity, and, further, that for positive values of x, the function $f(x)$ is always positive.

Hence it follows, from the equation of motion, that

$$\frac{du}{d\theta} = 0, \text{ and } r = c,$$

or that, with the assigned conditions, a circle is the only possible path.

If in this case a slight disturbance of path take place, a nearly circular orbit will be described, the apsidal angle being

$$\pi/\sqrt{n+3}.$$

118. *Motion of a particle under the action of a central force, in a medium the resistance of which varies as the square of the velocity.*

Taking the transversal acceleration, we have

$$\frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = -k\dot s^2 \frac{rd\theta}{ds} = -k\dot s r\dot\theta,$$

or

$$\frac{1}{r^2\dot\theta}\frac{d}{dt}(r^2\dot\theta) = -k\dot s,$$

and therefore $r^2\dot\theta = h\epsilon^{-ks}$, h being a constant; or, if p be the perpendicular on the tangent,

$$p\dot s = h\epsilon^{-ks}.$$

Taking the normal acceleration,

$$\frac{\dot s^2}{\rho} = P\frac{p}{r}, \text{ or } \frac{h^2}{p^2}\epsilon^{-2ks} = Pp\frac{dr}{dp}.$$

By help of the equation

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

this is transformed into

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}\,\epsilon^{2ks}.$$

119. *A particle is moving under the action of a central force in a resisting medium; it is required to find the resistance necessary for the description of a given path.*

Taking transversal and normal accelerations, and assuming mR as the force of resistance,

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=-R\frac{p}{r},\quad \frac{\dot{s}^2}{\rho}=P\frac{p}{r};$$

therefore $$\frac{d}{dt}(p\dot{s})=-Rp,\ \text{and}\ \frac{d}{dt}(p\dot{s})^2=-2Rp^2\dot{s},$$

or $$R=-\frac{1}{2p^2}\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right).$$

120. The two following problems will further illustrate the use of radial and transversal components, and the application of the principle of relative accelerations.

Two equal particles are attached to the middle point, and to the end C, of a string ABC, the end A of which is fixed on a smooth horizontal plane; it is required to find the equations of motion of the particles on the plane.

If θ and ϕ be the inclinations of AB and BC to a fixed line on the plane, the accelerations of B along BA and perpendicular to it are $a\dot{\theta}^2$ and $a\ddot{\theta}$, and the accelerations of C relative to B along CB and perpendicular to it are $a\dot{\phi}^2$ and $a\ddot{\phi}$. Compounding these accelerations perpendicular and parallel to CB, we obtain the equations of motion of C,

$$a\ddot{\phi}+a\ddot{\theta}\cos(\phi-\theta)+a\dot{\theta}^2\sin(\phi-\theta)=0\ \ldots\ldots(1),$$

$$a\dot{\phi}^2+a\dot{\theta}^2\cos(\phi-\theta)-a\ddot{\theta}\sin(\phi-\theta)=T'\ldots\ldots(2),$$

and, for the motion of B,

$$a\ddot{\theta}=T'\sin(\phi-\theta)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$a\dot{\theta}^2=T-T'\cos(\phi-\theta)\ \ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating T' we obtain,

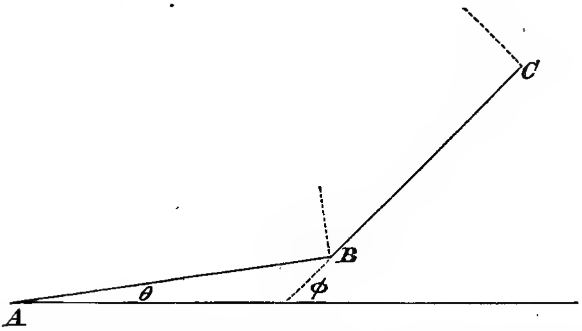
$$a\dot{\phi}^2\sin(\phi-\theta)+a\dot{\theta}^2\sin(\phi-\theta)\cos(\phi-\theta)-a\ddot{\theta}\{1+\sin^2(\phi-\theta)\}=0) \qquad \ldots\ldots\ldots\ldots(5).$$

Multiplying (1) by $1+\cos(\phi-\theta)$, and subtracting (5),

$$\ddot{\theta}+(\ddot{\phi}+\ddot{\theta})\{1+\cos(\phi-\theta)\}-(\dot{\phi}^2-\dot{\theta}^2)\sin(\phi-\theta)=0,$$

and therefore

$$\dot{\theta}+(\dot{\phi}+\dot{\theta})\{1+\cos(\phi-\theta)\}=C.$$



This is really the equation expressing the constancy of the angular momentum, and could have been written down at once.

Again, eliminating $\dot{\theta}^2$ from (1) and (5),

$$\dot{\phi}^2\sin(\phi-\theta)-\ddot{\phi}\cos(\phi-\theta)-2\ddot{\theta}=0 \ldots(6).$$

Multiplying (1) by $2\dot{\phi}$, (6) by $2\dot{\theta}$, and subtracting,

$$2\dot{\phi}\ddot{\phi}+4\dot{\theta}\ddot{\theta}+2(\dot{\theta}\ddot{\phi}+\dot{\phi}\ddot{\theta})\cos(\phi-\theta)-2(\dot{\phi}^2\dot{\theta}-\dot{\theta}^2\dot{\phi})\sin(\phi-\theta)=0,$$

and therefore

$$\dot{\phi}^2+2\dot{\theta}^2+2\dot{\theta}\dot{\phi}\cos(\phi-\theta)=D,$$

which is the equation expressing the constancy of energy, and could have been written down at once by observing that the velocity of C is compounded of $a\dot{\phi}$ perpendicular to CB, and of $a\dot{\theta}$ in the direction perpendicular to BA.

We have given the preceding solution for the sake of illustrating methods and principles, but so far as the present problem is concerned its solution is perhaps more easily effected by employing rectangular components. The equations are then, if x, y be co-ordinates of B, and ξ, η of C,

$$m\ddot{x} = T' \cos\phi - T\cos\theta, \quad m\ddot{y} = T' \sin\phi - T\sin\theta,$$

$$m\ddot{\xi} = -T' \cos\phi, \qquad m\ddot{\eta} = -T' \sin\phi,$$

with the geometrical conditions,

$$x = a\cos\theta, \quad \xi = a\cos\theta + a\cos\phi,$$

$$y = a\sin\theta, \quad \eta = a\sin\theta + a\sin\phi.$$

Multiplying the equations of motion by $2\dot{x}$, $2\dot{\xi}$, &c., and adding, we obtain

$$\dot{x}^2 + \dot{y}^2 + \dot{\xi}^2 + \dot{\eta}^2 = C;$$

we also find that

$$x\ddot{y} - y\ddot{x} + \xi\ddot{\eta} - \eta\ddot{\xi} = 0,$$

leading to

$$x\dot{y} - y\dot{x} + \xi\dot{\eta} - \eta\dot{\xi} = D.$$

These are the equations of energy and momentum, and by means of the geometrical equations are at once expressible in terms of θ and ϕ as before.

Of course the simplest solution consists in utilizing the principles of energy and of angular momentum, and at once writing down the two equations derived from these principles.

121. *The same system being suspended from the end A, it is required to determine the small oscillations in a vertical plane.*

Neglecting the squares of small quantities, and taking θ and ϕ as the inclinations to the vertical, the acceleration of C relative to B is $a\ddot{\phi}$ perpendicular to CB, and that of B is $a\ddot{\theta}$ perpendicular to BA. If T and T' be the tensions, the approximate equations of motion, neglecting the product

$\ddot{\theta}\sin(\phi-\theta)$, and taking $\cos(\phi-\theta)=1$, and $\sin(\phi-\theta)=\phi-\theta$, are

for C,
$$\left.\begin{aligned} m(a\ddot{\phi}+a\ddot{\theta}) &= -mg\phi \\ 0 &= -mg+T' \end{aligned}\right\},$$

and for B,
$$\left.\begin{aligned} ma\ddot{\theta} &= -mg\theta+T'(\phi-\theta) \\ 0 &= -mg+T-T' \end{aligned}\right\}.$$

We hence obtain $T'=mg$, and $T=2mg$;

therefore $a\ddot{\phi}=g(2\theta-2\phi)$, and $a\ddot{\theta}=g(\phi-2\theta)$.

Multiplying by λ and adding

$$a(\ddot{\phi}+\lambda\ddot{\theta})=-g(2-\lambda)\left(\phi+\frac{2\lambda-2}{2-\lambda}\theta\right),$$

and if we assume $\dfrac{2\lambda-2}{2-\lambda}=\lambda$, we find that $\lambda=\pm\sqrt{2}$.

We hence obtain two equations of the form

$$\ddot{\phi}+\lambda\ddot{\theta}+\frac{g}{a}(2-\lambda)(\phi+\lambda\theta)=0,$$

the solution of each being of the form

$$\phi+\lambda\theta=A\cos nt+B\sin nt, \text{ Chapter II.},$$

shewing that the values of θ and ϕ are each compounded of harmonic quantities.

EXAMPLES.

1. A heavy particle is fastened by two equal strings to two points in a horizontal line, and then whirled round in a vertical plane; the velocity is such that if one of the strings break when the particle is at its lowest point or when it is half-way between its highest and lowest points, the particle will continue to describe a circle; find the least distance between the points of suspension that this may be possible.

2. If a body move in an ellipse under a force to a focus, the velocity at the mean distance from the centre of force is a mean proportional between the velocities at the extremities of any diameter.

3. A particle describes an ellipse about a centre of force at a vertex A. Prove that the force varies as $AP \div AN^3$, where N is the foot of the perpendicular from P let fall on the major axis.

4. Find the law of force to the pole when the path is the cardioid, $r = a(1 - \cos\theta)$; and prove that, if F be the force at the apse, and v the velocity,

$$3v^2 = 4aF.$$

5. If a particle be describing an ellipse about a centre of force in the centre, shew that the sum of the reciprocals of its angular velocities about the foci is constant.

6. A particle is describing the ellipse $l = r(1 + e\cos\theta)$ under the action of a force tending to the origin; if the velocity be altered at any point so as to make it describe a parabola, shew that the vertex of the parabola lies on the curve

$$r = a(1 - e\cos\theta).$$

7. A particle describes an ellipse about a central force in the focus S, SY is a perpendicular upon the tangent to the orbit; shew that the angular velocity of SY is a minimum or maximum when the particle is at the farther apse according as the eccentricity is less or greater than $1/3$.

8. A particle is acted upon by a central force which is some function of the distance (r) and by a transverse force equal to μr. Shew that, if the particle move completely round any closed curve, the square of the velocity is increased by 4μ (area of the curve).

9. A particle describes a lemniscate about a centre of force in the node; shew that the velocity of the corresponding point of the rectangular hyperbola to which the lemnis-

cate is related varies as the 5th power of the radius vector of that point.

10. A point moves in a plane curve and sounds as it moves. At a fixed point C in the plane the whole sound produced is heard simultaneously. Shew (i) that if the point moves uniformly, the curve is an equiangular spiral—(ii) if the point moves as in a central orbit about C, the curve is a reciprocal spiral.

11. Prove that the acceleration of a particle describing an epicycloid under a centre of force at the centre of the fixed circle varies as r/p^4.

12. If at any point of a parabolic orbit about the focus, the velocity be diminished in a given ratio, prove that the empty foci of the elliptic orbits corresponding to different points of change lie in a parabola.

13. An imperfectly elastic particle is under the influence of a smooth hard gravitating sphere. Shew that (excepting special circumstances of projection) it will perpetually describe arcs of conic sections: determine also the elements of the orbit described after any number of rebounds.

14. From every point of an ellipse particles are projected in the direction of the tangent with velocities such that, when acted on by a centre of force μ/r^2 in one of the foci of the ellipse, they proceed to describe parabolas. Shew that the directrices of these parabolas all touch one or other of two fixed circles whose radii are equal to the major axis of the given ellipse.

15. If v_1, v_2 be the velocities at the extremities of a diameter of an ellipse described about the focus, and u the velocity at either of those points when it is described about the centre, prove that $u\,(v_1 + v_2)$ is constant.

16. Two equal and perfectly elastic particles are under the action of the same centre of force μ/r^2; the one is describing an ellipse and the other a confocal hyperbola, the semi-major axes being (a), (a') respectively. If they impinge

then after impact they will describe two conics, cutting each other orthogonally and of semi-major axes (α, α'), where

$$\frac{1}{a}+\frac{1}{a'}=\frac{2}{a+a'}\pm\left(\frac{2}{a+a'}+\frac{1}{a}-\frac{1}{a'}\right),$$

the upper or lower sign being taken according as the hyperbola is described about the outer or inner focus.

17. OY is the perpendicular from a fixed point O on the tangent to a curve at any point P. If the curve is such that PY is constant, and a particle describe it under the action of a force to O, prove that the force varies as $OP \div OY^4$.

18. A number of circles touch at a point P, and particles describe them under forces to a point S, on the line through P perpendicular to the common tangent, inside all the circles, the forces on all the particles when at P being the same: prove that the squares of the periodic times vary as the cubes of the radii of the circles.

19. A curve described by a particle under the action of a central force is such that, if at any moment the component velocity along the radius vector be destroyed by an impulse along the radius vector, the particle will proceed to describe a circle: prove that the curve is a reciprocal spiral.

20. The ends of a straight tube AB, of length $4a$, are connected by an elastic string, the natural length of which is $2a$; a particle is fastened to the middle point of the string, and the tube is then made to move uniformly, in a horizontal plane, about a fixed point O, with which it is rigidly connected, and which is equidistant from its ends: determine the motion of the particle, examining the different cases which may occur in the question.

21. A particle is placed in a straight tube which revolves uniformly in a vertical plane about its lowest end. Supposing the particle to have no initial velocity relatively to the tube and that initially the free end of the tube is vertically above the fixed end, prove that the velocities of the particle along the tube, when the tube returns successively to its initial position, are proportional to

$$e^{2\pi}-e^{-2\pi},\ e^{4\pi}-e^{-4\pi},\ e^{6\pi}-e^{-6\pi},\ldots\ldots$$

22. Prove that the law of force under which the pedal of $p = f(r)$ can be described is

$$h^2 \left\{ \frac{2r^2}{p^5} - \frac{r}{p^4} \frac{dr}{dp} \right\}.$$

If $p \propto r^n$ the law of force under which the pedal can be described varies inversely as $(\text{distance})^{5-\frac{2}{n}}$.

23. Prove that, if a particle move in a smooth tube under the action of any forces tending to centres, the pressure at any point of the tube will vary as

$$\frac{1}{\rho} \left\{ C - \Sigma \frac{1}{p} \frac{d}{dp} (p^2 F) \right\},$$

where $\frac{dF}{dr}$ is the acceleration towards any one of the centres, and ρ is the radius of curvature; and hence, that the pressure at any point of the tube will vary as the curvature, whenever the orbit is such as could be described freely under the action of each of the forces taken separately.

24. A particle moves in a plane under forces whose resultant in any direction is $\frac{dV}{ds}$, where ds is an element of the line drawn in that direction from the position of the particle; prove that the polar equation of the orbit is

$$h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u \right) = u^2 \frac{dV}{du} - \frac{du}{d\theta} \frac{dV}{d\theta}.$$

In case $V = \mu u^2 \cos\theta$, and the particle be projected initially so that $\theta = 0$, $h^2 = 2\mu$, find the orbit and shew that between $\theta = 0$ and $\theta = \frac{\pi}{2}$, the radius vector is described with uniform velocity.

25. The attraction to a given point, at a distance r, $= 3\mu/r^3 + 2\mu a^2/r^5$. A particle is projected in a direction making an angle $\tan^{-1}\frac{1}{2}$ with the initial distance (a), and with a velocity equal to that in a circle at the same distance; determine the orbit described.

26. A body is describing an ellipse of eccentricity $\sqrt{\frac{7}{8}}$ under a force to the focus S: when the particle is at one end B of the minor axis, the centre of force is suddenly transferred to a point S' in BS produced such that $BS' = 4BS$, and the absolute force is doubled and becomes repulsive: prove that the new orbit is a rectangular hyperbola.

27. If the orbit be $r = a \sin n\theta$, shew that the attraction is

$$h^2 \left[\frac{2n^2a^2}{r^5} - \frac{n^2-1}{r^3}\right].$$

28. Force $\propto 2au^2 + 9u^3 + 6au^4$. A body falls freely from infinity towards the centre of force till its distance is a, and then its direction is suddenly turned through the angle $\cot^{-1} 4$; find the orbit described.

29. A particle is tied by an elastic string of length a to a point wherein resides a repulsive force $\propto$ (dist.)^2: its initial distance is a, and it is projected at right angles to this with a velocity $= \sqrt{3}/2$ that necessary for circular motion, were the force attractive. If it passes through a point of inflection at a distance $3a/2$, shew that it will come to a second apse at the distance $3a$.

30. Having given $P = \mu u^5$, and that a particle is projected from an apse at the distance c, find the orbit (1) when the velocity of projection is $\sqrt{\mu}/c^2\sqrt{2}$, (2) when it is $\sqrt{\mu}/c^2$.

31. SN is a fixed line through a centre of force S; PN is the ordinate at any point P of the path of a particle acted on by the central force; find the force when the curve PS always bisects the triangle PNS.

32. If the path be the lemniscate the Bernoulli, the equation to which is $r^2 = a^2 \cos 2\theta$, prove that the square of the force varies as the seventh power of the angular velocity of the radius vector of the particle.

33. A point is moving in an equiangular spiral, its acceleration always tending towards the pole, S. When it arrives at a point P, the law of the acceleration is changed

to that of the direct distance, the actual acceleration at P being unaltered. Prove that the point will now move in an ellipse whose axes make equal angles with SP and the tangent to the spiral at P, and that the ratio of these axes is $\tan\frac{\alpha}{2} : 1$, where α is the angle of the spiral.

34. A particle m is projected at a distance (a) from a fixed point, in a direction at right angles to that distance, with a velocity $\sqrt{\mu}/2a\sqrt{2}$, and is acted upon by a repulsive force $m\mu/r^3$ from that point; find its path, and prove that the time from the distance a to the distance $a\sqrt{2}$ is $2\sqrt{2}a^2/3\sqrt{\mu}$.

35. A particle moves under a force $\mu\,\{3au^4 - 2\,(a^2 - b^2)\,u^5\}$, a being $> b$, and is projected from an apse at a distance $a + b$ with velocity $\sqrt{\mu} \div (a + b)$: shew that its orbit is

$$r = a + b\cos\theta.$$

36. If the parabolic orbits of two comets intersect the orbit of the earth, supposed circular, in the same two given points, and if t_1, t_2 be the times in which the comets respectively move from one of these points to the other, prove that

$$(t_1 + t_2)^{\frac{2}{3}} + (t_1 - t_2)^{\frac{2}{3}} = \left(\frac{4}{3\pi}\right)^{\frac{2}{3}},$$

the unit of time being a year.

37. The pedal of a curve with respect to a point is defined to be the locus of the foot of the perpendicular drawn from the point on any tangent to the curve, and the pedal of the pedal with respect to the same point is called the second pedal, and so on; a particle describes the n^{th} pedal freely under the action of a force tending to that point; find the law of force.

If the curve be a rectangular hyperbola, and the pedals be formed with respect to its centre, prove that the n^{th} pedal will be the orbit of a particle moving under the action of a force varying as $r^{\frac{6n+1}{2n-1}}$, where r is its distance from the centre of force.

38. A particle is projected with velocity v from the vertex of a cycloid and describes the curve under an attraction to a centre of force situated on the axis at a distance from the vertex greater than the diameter of the generating circle and less than twice that diameter; prove that the particle will be again at an apse after a time

$$\frac{a}{v}\cdot\frac{2\alpha^2\cos\alpha+3\alpha\sin\alpha+3\sin^2\alpha\cos\alpha}{\alpha\cos\alpha+\sin\alpha},$$

a being the radius of the generating circle of the cycloid, and α the apsidal angle.

39. A particle P describes a central orbit, centre of force S, and through P is drawn a straight line at right angles to PS, which line touches its envelope in Q: prove that the velocity of Q

$$\propto \frac{1}{r^2}\left(\frac{d^2r}{d\theta^2}+r\right);$$

and is constant only when the path of P is a parabola, whose focus is S.

40. A particle is projected at a distance a with a velocity equal to the velocity in a circle at the same distance, and at an angle $\pi/4$ with distance, the force being $\mu(3r^5+a^2r^3)/r^5$; determine the orbit described, and shew that the time to the centre of force is $(4-\pi)\,a^2/2\sqrt{2}\mu$.

41. A particle is projected from an apse with the velocity from infinity under the attraction of a force $\frac{\mu}{r^3}\log\frac{r}{a}$ directed to a centre at a distance a: find the equation of the orbit described.

42. A point describes a semi-ellipse, bounded by the conjugate axis, and its velocity, at a distance r from the focus, is $a\left\{\frac{f(a-r)}{r(2a-r)}\right\}^{\frac{1}{2}}$, $2a$ being the length of the transverse axis, and f a constant acceleration; prove that the acceleration of the point is compounded of two, each varying inversely as the square of the distance, one tending to the nearer focus, and the other from the farther focus.

43. If the force at a distance r be $2\mu\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)$ and the particle be projected from an apse at a distance a with a velocity $\frac{\sqrt{\mu}}{a}$, it will be at a distance r from the centre after a time

$$\frac{1}{2\sqrt{\mu}}\left\{a^2\log\frac{r+\sqrt{r^2-a^2}}{a}+r\sqrt{r^2-a^2}\right\}.$$

44. In an orbit described under the action of a central force, a straight line is drawn from the centre of force perpendicular to the tangent and proportional to the acceleration: if this straight line describes equal areas in equal times, shew that the equation to the orbit is of the form

$$\frac{1}{p^{\frac{4}{3}}}=\frac{1}{c^{\frac{4}{3}}}\pm\left(\frac{r}{a^2}\right)^{\frac{4}{3}}.$$

Shew that the rectangular hyperbola is a particular case.

45. Having given $P=5\mu u^3+8\mu c^2u^5$, and that a particle is projected from an apse at the distance c with the velocity $3\sqrt{\mu}/c$, prove that the orbit is $r=c\cos 2\theta/3$.

46. A particle acted on by the central force $\mu(r+a)/r^3$ is projected from an apse at a distance a and with a velocity which is to that in a circle at the same distance as $1:\sqrt{2}$; shew that the equation to the orbit is $r(2+\theta^2)=2a$, and that the particle will arrive at the pole in time $\pi\sqrt{a^3}/\sqrt{8\mu}$.

47. Force $\propto(8a^3u^5-12a^2u^4+9au^3-2u^2)$. A particle is projected at distance a with a velocity $=\sqrt{2/3}$ that in a circle at the same distance, and in a direction making an angle $\pi/4$ with the radius vector; find the orbit described, and prove that its equation is $r=a(1-\tan\theta)$.

48. Prove that there are two directions in which a particle, acted upon by a central force varying as the distance, can be projected from a given point, with given velocity, so as to pass through another given point.

49. The attraction to a fixed point varying inversely as the fifth power of the distance, a particle is projected in

a direction making the angle $\tan^{-1} 2\sqrt{2}/3$ with the initial distance c, with the velocity $\sqrt{17\mu}/\sqrt{2c^2}$; prove that the orbit is

$$\frac{2r}{c} = \frac{3\epsilon^{\theta\sqrt{2}} - 1}{3\epsilon^{\theta\sqrt{2}} + 1}.$$

50. Two heavy particles are connected by a string without weight. One particle is just dropped through a hole in a smooth horizontal plane and the other is projected on the plane at right angles to the string fully stretched.

(1) Find the least velocity of projection which will keep the particle from descending.

(2) If the velocity of projection be less than this, determine the motion of the descending particle.

51. A particle describes an equiangular spiral under the action of a central force to the pole in a medium in which the resistance varies as the square of the velocity. Prove that the distance from the pole at which the central force is a maximum is half the distance at which the velocity is a maximum, and that these distances are independent of the initial distance or initial velocity.

52. Two equal particles P and Q, on a smooth horizontal plane, are connected with each other by a stretched inelastic string of length l, which passes through a smooth ring (O) fixed to the plane; P being projected perpendicularly to PQ, find the equation to its path, and shew that, when Q arrives at (O), P will have described a right angle about (O) if the initial distance of P from it be equal to $l \cos \pi/2\sqrt{2}$.

53. A rod which is extensible in accordance with Hooke's law is made to rotate with uniform angular velocity ω, about a line through one extremity perpendicular to its length, and is supposed to remain straight. If a be its natural length, m the mass of unit of length when unstretched, E its modulus of elasticity, and l its length when rotating, shew that

$$l = \frac{1}{\omega}\sqrt{\frac{E}{m}} \tan\left\{a\omega\sqrt{\frac{m}{E}}\right\},$$

that the stress on the axis is

$$E\left\{\sec a\omega\sqrt{\frac{m}{E}}-1\right\},$$

and that the rod must break before its angular velocity is $\pi\sqrt{E}/m\sqrt{2a}$.

54. A wire in the form of a plane curve is constrained to rotate about an axis perpendicular to its plane with varying angular velocity. Find the motion of a bead which slides upon it under the action of any given forces, and the pressure on the wire.

If the wire is circular, and the axis through a point in its circumference, and the angular velocity ω uniform, shew that the pressures on the curve at the two extremities of the diameter perpendicular to that through the axis are $(3 \pm 2\sqrt{2})\,m\omega^2 a$, the particle starting from rest at a point near the axis. Draw a figure to indicate at which of these points the pressure is the greater, and the direction of that pressure.

55. A small smooth ring can move upon a smooth circular wire which is made to roll with uniform angular velocity ω on the outside of a horizontal circle of n times its radius: prove that the angle through which the ring will have revolved with respect to the wire at a time t is

$$\{(n+1)\,\phi - n\omega t\}/(n+1) \text{ where } \ddot{\phi} = \omega^2 \sin\phi/(n+1).$$

56. Two particles P, Q, of equal mass, slide upon a smooth endless string OPQ, which passes through a small smooth ring at O, and lies on a smooth horizontal plane. OP is initially equal to OQ, and the particles are projected with equal velocities along the external bisectors of the angles OPQ, OQP respectively; prove that, throughout the motion, the tension of the string varies inversely as OP.

57. A particle of mass m is attached to a fixed point by an elastic string of natural length a, and whose coefficient of elasticity is m. It is projected with the velocity due to half the length of the string in a direction perpendicular to the

string which is initially unstretched. Prove that the apsidal distances of its orbit are given by

$$a^2(r^2-a^2)-r^2(r-a)^2=0.$$

58. A body is moving in a uniform resisting medium, the resistance of which varies as the n^{th} power of the velocity, under the action of a force varying inversely as the square of the distance; find the value of n in order that the path may be an equiangular spiral.

59. Two centres of force μr, $\mu' r'$, tend to two fixed points S, S', the former being attractive and the latter repulsive. Prove that a particle under their influence moves in such a way that

$$\mu r^2\frac{d\theta}{dt}+\mu' r'^2\frac{d\theta'}{dt}=c^2,$$

c being a constant and θ, θ' the angles that r, r' make with SS'.

60. A smooth circular tube is fixed at one point A and contains a particle which is initially at rest at the opposite extremity of the diameter through A. The tube is then made to revolve in its own plane with a uniform angular velocity ω; shew that the angle described by the particle about the centre of the tube in the time t is

$$4\tan^{-1}\left(\frac{\epsilon^{\omega t}-1}{\epsilon^{\omega t}+1}\right).$$

61. A heavy particle is attached to a fixed point by means of an elastic string of natural length $3a$, whose coefficient of elasticity is six times the weight of the particle; when the string is at its natural length and the particle vertically above the point of attachment the particle is projected horizontally with a velocity $3\sqrt{ag}/\sqrt{2}$; prove that the angular velocity of the string will be constant and that the particle will describe the curve $r=a(4-\cos\theta)$.

62. Prove that the central force tending to the pole which causes a particle to describe an equiangular spiral in a homogeneous medium wherein the resistance varies as the

n^{th} power of the velocity (the initial velocity having been properly chosen) varies inversely as $(r)^{\frac{n}{n-2}}$ where r is the distance of any point from the pole. Find also the velocity at any point in terms of r. Examine the cases where $n = 2$ or 1.

63. Shew that a particle may be made to describe an epicycloid under the attraction of a circular wire whose particles attract inversely as the cube of the distance, the circular wire being the fixed circle of the epicycloid.

64. A heavy particle is free to move on a smooth elliptic wire which is moveable in a vertical plane about its centre. The wire is placed with its major axis vertical and the particle at the highest point. The wire is then suddenly started to rotate with a constant angular velocity $\sqrt{\dfrac{g}{2(a+b)}}$, where a, b are the semi-axes of the ellipse. Shew that the particle will move relatively to the ellipse as if it were acted upon by a force to the centre varying as the distance.

65. A particle m is projected from an apse at distance c with velocity $\sqrt{2\mu c^3}/\sqrt{3}$. The force to the centre being $m\mu(r^5 - c^4 r)$, prove that the orbit is $x^4 + y^4 = c^4$.

66. When the forces have a potential equal to $\mu r^{-2}\cos\theta$ and a particle is projected at a distance a perpendicular to the initial line with velocity $2\sqrt{\mu}/a$, then the orbit described is

$$r = a\sec\left(\sqrt{2}\log\tan\frac{\pi+\theta}{4}\right).$$

67. A particle acted upon by a constant repulsive central force, is projected at right angles to the initial distance with a velocity double that which would be required in moving from the centre of force to the point of projection; prove that the orbit is

$$\frac{\theta}{2} = \tan^{-1}\sqrt{\frac{r-a}{a}} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{r-a}{3a}}.$$

68. A smooth circular wire rotates uniformly in its own plane, which is horizontal, about a fixed point, the distance of which from its centre is one-third of its radius, and a bead which can move on the wire is attached to it at the point nearest the fixed point. If the bead be set free, prove that it will make complete revolutions, and that, at the angular distance $\sec^{-1} 3$, its pressure on the wire will vanish.

69. A particle is projected towards the origin from infinity with any velocity, and is acted upon by a force μu^3 at right angles to the radius vector; shew that it will describe a curve of the family, $u = a\theta^{\frac{1}{4}} J_{\frac{1}{4}}(\theta)$, where $J_n(x)$ is the Bessel's function of the n^{th} order, and find the velocity of projection in order that a particular curve may be described.

CHAPTER VIII.

TANGENTIAL AND NORMAL ACCELERATIONS.

122. If mT and mN be the forces acting on a particle of mass m in directions of the tangent and normal, the equations of motion are

$$m\ddot{s} = mT, \text{ or } mv\frac{dv}{ds} = mT,$$

and
$$m\frac{v^2}{\rho} = mN.$$

We have already made a slight use of the expression for normal acceleration in article (96); we now proceed to develope, somewhat at length, the utilization of these expressions.

Motion of a heavy particle on a smooth curve in a vertical plane.

Measuring x horizontally, and y vertically downwards, and taking R as the normal reaction of the curve, measured outwards, the equations of motion are

$$mv\frac{dv}{ds} = mg\frac{dy}{ds}, \quad m\frac{v^2}{\rho} = mg\frac{dx}{ds} - R.$$

The first equation gives

$$\tfrac{1}{2}m(v^2 - u^2) = mg(y - y'),$$

if u be the velocity when $y = y'$,

This is, in effect, the equation of energy, and can be written down at once from the assumption of the truth of the principle of energy.

The second equation determines the pressure.

123. *Motion of a particle on a smooth curve under the action of forces to fixed centres, the forces being functions of the distances from those centres.*

If r, $r',\ldots$ be the distances of the particle from the centres of force, and mP, $mP'\ldots$ the forces, the equations of motion are

$$mv\frac{dv}{ds} = mP\frac{dr}{ds} + mP'\frac{dr'}{ds} + \ldots$$

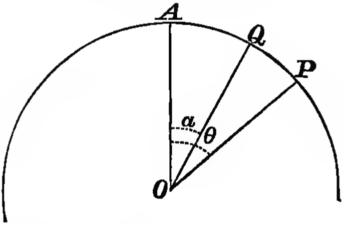
$$m\frac{v^2}{\rho} = mPr\frac{d\theta}{ds} + mP'r'\frac{d\theta'}{ds} + \ldots + R.$$

From the first,

$$\tfrac{1}{2}mv^2 = \Sigma\int mPdr;$$

this, which is the equation of energy, gives the velocity and the second equation determines the pressure.

124. *Motion of a heavy particle, placed on the outside of a smooth circle and allowed to slide down.*



If the particle start from the point Q, at an angular distance α from the vertex, and v be the velocity at P,

$$v^2 = 2ga(\cos\alpha - \cos\theta),$$

and $\frac{mv^2}{a} = mg\cos\theta - R$, R being the outward reaction of the curve,

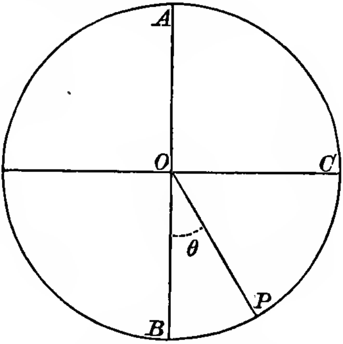
$$\therefore\ R = mg\,(3\cos\theta - 2\cos\alpha),$$

shewing that the pressure vanishes, and that the particle flies off the curve, when

$$\cos\theta = \tfrac{2}{3}\cos\alpha.$$

125. *Motion of a heavy particle inside a smooth circular tube in a vertical plane.*

We shall suppose that the particle starts with a given velocity u from the lowest point B.



Measuring θ upwards, the equation of energy is

$$\tfrac{1}{2}m\,(v^2 - u^2) = -mga\,(1 - \cos\theta),$$

and the equation for the pressure is

$$\frac{mv^2}{a} = R - mg\cos\theta,$$

R being the pressure, inwards, of the tube on the particle.

$$\therefore\ \frac{R}{m} = 3g\cos\theta - 2g + \frac{u^2}{a}.$$

To find the height of ascent we put $v=0$, and to find the point where the pressure vanishes we put $R=0$.

(1) Take $u^2<2ga$; then the highest point is given by $2ga\cos\theta=2ga-u^2$, and the pressure never vanishes.

(2) If $u^2=2ga$, the particle rises to C and the pressure then vanishes.

(3) If $u^2>2ga$ and $<4ga$, the highest point is given by

$$\cos\theta=-\frac{u^2-2ga}{2ga},$$

and the pressure vanishes, and changes sign, when

$$\cos\theta=-\frac{u^2-2ga}{3ga}.$$

(4) If $u^2>4ga$ and $<5ga$, the particle rises to A and passes over and the pressure vanishes when

$$\cos\theta=-\frac{u^2-2ga}{3ga}.$$

(5) If $u^2=5ga$, the pressure vanishes at A.

(6) If $u^2>5ga$, the pressure never changes sign.

Oscillation of a Pendulum.

126. A heavy particle, suspended by a string from a fixed point, and oscillating in a vertical plane, forms a simple pendulum.

Measuring θ from the vertical, and observing that if a be the length of the string, $s=a\theta$, the equation of motion is

$$a\ddot{\theta}=-g\sin\theta.$$

If the amplitude of oscillation be very small, the approximate equation is

$$\ddot{\theta}+\frac{g}{a}\theta=0,$$

and therefore $$\theta=A\cos\left(\sqrt{\frac{g}{a}}\,t+a\right).$$

This represents isochronous vibrations, the time of a complete vibration being $2\pi\sqrt{\frac{a}{g}}$.

Finite motion of a Pendulum.

Recurring to the equation, $a\ddot{\theta} = -g\sin\theta$, and supposing the pendulum to start at an inclination α to the vertical, we obtain

$$a\dot{\theta}^2 = 2g(\cos\theta - \cos\alpha),$$

and therefore if τ be the time of oscillation, from one side to the other,

$$\tau = 2\sqrt{\frac{a}{2g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{a}{g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}.$$

Putting $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\psi$, this transforms into

$$\tau = 2\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\psi}}$$

$$= 2\sqrt{\frac{a}{g}}F\left(\sin\frac{\alpha}{2}, \frac{\pi}{2}\right).$$

If θ be the angle at the time t from the lowest point,

$$t = \sqrt{\frac{a}{2g}}\int_0^{\theta}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{a}{g}}\int_0^{\psi}\frac{d\psi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\psi}}$$

$$= \sqrt{\frac{a}{g}}F\left(\sin\frac{\alpha}{2}, \psi\right),$$

or, in the notation of Jacobi and Guderman,

$$\psi = am\left(\sqrt{\frac{g}{a}}\,t\right),$$

and the height of the bob of the pendulum $= 2a\sin^2\frac{\theta}{2}$

$$= 2a\sin^2\frac{\alpha}{2}\;sn^2\left(\sqrt{\frac{g}{a}}\,t\right),\; mod.\sin\frac{\alpha}{2}.$$

127. *Motion of a heavy particle on the arc of a smooth cycloid, having its vertex downwards, and axis vertical.*

Measuring ϕ from the tangent at the vertex, the intrinsic equation of a cycloid is

$$s = 4a\sin\phi,$$

and the equation of motion is

$$\ddot{s} = -g\sin\phi, \text{ or } \ddot{s} + \frac{g}{4a}s = 0;$$

$$\therefore s = A\cos\left(\sqrt{\frac{g}{4a}}\,t + \alpha\right),$$

shewing that the motion is an isochronous vibration, the period of a complete vibration being

$$4\pi\sqrt{\frac{a}{g}}.$$

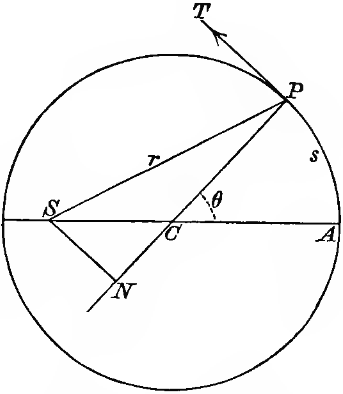
128. *Motion of a particle in a smooth circular tube under the action of a force to a fixed point varying as the distance from that point.*

Take a as the radius of the circle, and c as the distance of the centre of force S from the centre C.

Resolving along the tangent, the equation of motion is

$$v\frac{dv}{ds} = \mu r\cos SPT = -\mu r\frac{dr}{ds},$$

and therefore $v^2 = \mu\{(c+a)^2 - r^2\} = 2\mu ac(1-\cos\theta)$, supposing the particle to start from A.



If R be the pressure on the particle, measured inwards,

$$\frac{mv^2}{a} = m\mu r\cos CPS + R = m\mu PN + R,$$

or $$ma\dot{\theta}^2 = m\mu(a + c\cos\theta) + R,$$

therefore $$R = m\mu(2c - a - 3c\cos\theta).$$

129. *Motion of a particle on the arc of a smooth equiangular spiral under the action of a force from the pole.*

If mP be the force from the pole, and R the pressure measured inwards, we have

$$v\frac{dv}{ds} = P\cos\alpha, \quad \frac{mv^2\sin\alpha}{r} = mP\sin\alpha - R.$$

In the case in which $P = \mu/r^2$ these equations give

$$v\frac{dv}{dr} = \frac{\mu}{r^2}, \quad \text{or, } v^2 = 2\mu\left(\frac{1}{a} - \frac{1}{r}\right),$$

if the particle start with no initial motion from the distance a, and the second equation determines the pressure.

We also have

$$\frac{dr}{dt} = v\cos\alpha = \cos\alpha\sqrt{\frac{2\mu(r-a)}{ar}},$$

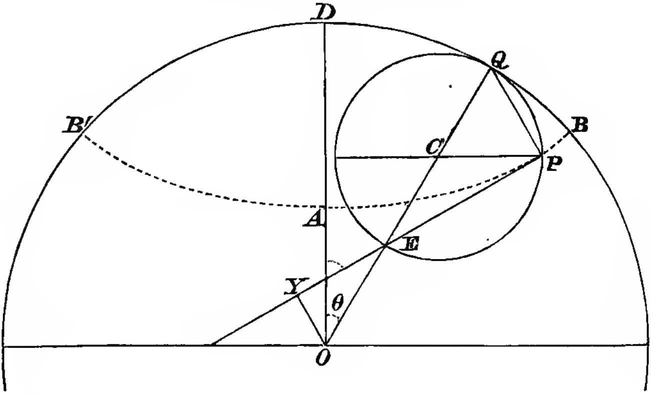
and therefore the time from the starting point to the distance b is equal to

$$\int_a^b \frac{\sqrt{ar}\,dr}{\cos\alpha\sqrt{2\mu(r-a)}}.$$

130. *Motion of a particle on the arc of a smooth hypocycloid, under the action of a force to the centre varying as the distance.*

Taking BAB' as the arc of the hypocycloid, of which A is the vertex, measure the arc from A.

In the figure C being the centre of the rolling circle of radius b, Q is the instantaneous centre, and EP, PQ are the tangent and normal to the hypocycloid at P.



The angle YOA being ϕ, we see that

$$\phi + \theta = \frac{a\theta}{2b} \text{ and } \therefore\ \theta = \frac{2b\phi}{a-2b},$$

and $$p = OY = (a-2b)\cos\frac{a\theta}{2b} = (a-2b)\cos\frac{a\phi}{a-2b}.$$

The curve being convex to the point O,

$$\frac{ds}{d\phi}=-p.-\frac{d^2p}{d\phi^2}=\frac{4b\,(a-b)}{a-2b}\cos\frac{a\phi}{a-2b},$$

and therefore $$s=4\frac{b}{a}(a-b)\sin\frac{a\phi}{a-2b}.$$

The equation of motion is

$$\ddot{s}=-\mu\,OP\cos OPY=-\mu PY=\mu\frac{dp}{d\phi}.$$

or, $$\ddot{s}+\frac{\mu a^2}{4b\,(a-b)}\,s=0,$$

shewing that the motion is oscillatory and isochronous.

A geometrical proof of the isochronism of the hypocycloid will be found in the Principia, Book I., Section X.

If we make the radius of the circle infinitely large, and the quantity μ infinitely small, and take $\mu a=g$, we fall upon the case of cycloidal motion, and the above equation becomes

$$\ddot{s}+\frac{g}{4b}\,s=0, \text{ as in Art (127).}$$

131. *Motion of a particle sliding on a rough curve.*

The equations of motion are, if R be the pressure, measured inwards, and μ the coefficient of friction,

$$mv\frac{dv}{ds}=mT-\mu R,$$

$$m\frac{v^2}{\rho}=mN+R.$$

Taking the case, for instance, of a heavy particle sliding upwards on the arc of a curve in a vertical plane, we have

$$mv\frac{dv}{ds}=-mg\sin\phi-\mu R,$$

$$m\frac{v^2}{\rho}=R-mg\cos\phi,$$

where ϕ is the inclination of the tangent to the horizon.

Hence $$v\frac{dv}{ds}+\mu\frac{v^2}{\rho}=-g(\sin\phi+\mu\cos\phi),$$

or, $$\frac{d\,.\,v^2}{d\phi}+2\mu v^2=-2g\rho(\sin\phi+\mu\cos\phi),$$

and, the intrinsic equation $s=f(\phi)$ being given, $\rho=f'(\phi)$, the equation is that of Chapter II. (1).

132. *Motion of a heavy particle in a medium the resistance of which varies as the square of the velocity.*

Measuring ϕ downwards from a horizontal line, the equations of motion are

$$v\frac{dv}{ds}=g\sin\phi-kv^2,\text{ and }\frac{v^2}{\rho}=g\cos\phi.$$

The second equation shews that for a given velocity the curvature is independent of the resistance, a theorem which is true for motion in any resisting medium under any forces.

Eliminating v we obtain

$$\cos\phi\frac{d\rho}{d\phi}-3\rho\sin\phi+2k\rho^2\cos\phi=0,$$

or, $$\frac{d}{d\phi}\left(\frac{1}{\rho}\right)+3\tan\phi\,.\,\frac{1}{\rho}=2k,\text{ leading to}$$

$$\frac{\sec^3\phi}{\rho}=k\{\tan\phi\sec\phi+\log(\tan\phi+\sec\phi)\}+C,$$

and this is the intrinsic equation of the path.

133. The question sometimes arises whether a given curve can be described by a particle under the action of forces to two or more fixed points. In such cases we obtain equations for $v\dfrac{dv}{ds}$ and $\dfrac{v^2}{\rho}$, and the value of v^2 obtained by integrating the first must be identified with the value of v^2 obtained from the second.

If for instance the curve be an ellipse and the forces $\dfrac{\mu}{r^2}$ and $\dfrac{\mu'}{r^2}$ to the two foci, we have

$$v\frac{dv}{ds}=-\frac{\mu}{r^2}\frac{dr}{ds}-\frac{\mu'}{r'^2}\frac{dr'}{ds},$$

$$\frac{v^2}{\rho}=\frac{\mu}{r^2}\sin\phi+\frac{\mu}{r'^2}\sin\phi.$$

From these equations,

$$v^2 = C + \frac{2\mu}{r} + \frac{2\mu'}{r'},$$

and $$v^2 = \frac{\mu}{r^2} \cdot \frac{rr'}{a} + \frac{\mu}{r'^2}\frac{rr'}{a} \text{ since } \rho \sin\phi = \frac{CD^2}{AC},$$

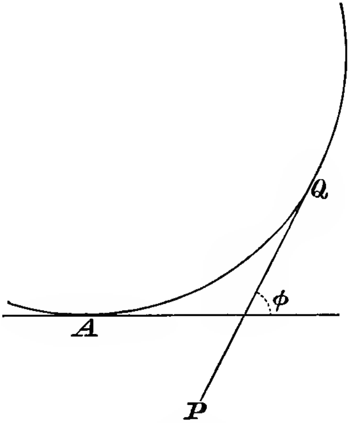
$$= \frac{2\mu}{r} + \frac{2\mu}{r'} - \frac{\mu}{a} - \frac{\mu'}{a},$$

and the values of v^2 are the same if

$$C = -\frac{\mu}{a} - \frac{\mu'}{a}.$$

134. *Motion of a particle fastened to a string which is wound round a fixed curve.*

Suppose that when the string is completely wound up, the particle is at A, and measure s and ϕ from A and the tangent at A.



Then $PQ = s$, and the path of P is an involute of the curve, its radius of curvature being PQ.

Taking mS and mN as the forces perpendicular and parallel to the string, and T as the tension, the equations of motion are

$$\frac{mvdv}{d\sigma} = mS, \quad m\frac{v^2}{\rho} = T + mN,$$

σ being an arc of the path, or

$$\frac{mvdv}{sd\phi} = mS, \quad m\frac{v^2}{s} = T + mN.$$

If there are no acting forces, v is constant, and the tension varies inversely as PQ.

Ex. *Let the curve be a circle and the acting force a repulsive force from the centre varying as the distance.*

In this case $$v\frac{dv}{d\sigma} = \mu a,$$

so that the motion is uniformly accelerated, and

$$\frac{mv^2}{s} = T - m\mu s,$$

so that, if the particle start from rest at A,

$$T = 2m\mu s.$$

135. *Motion of a string or fine chain inside a smooth tube of any shape under the action of given forces.*

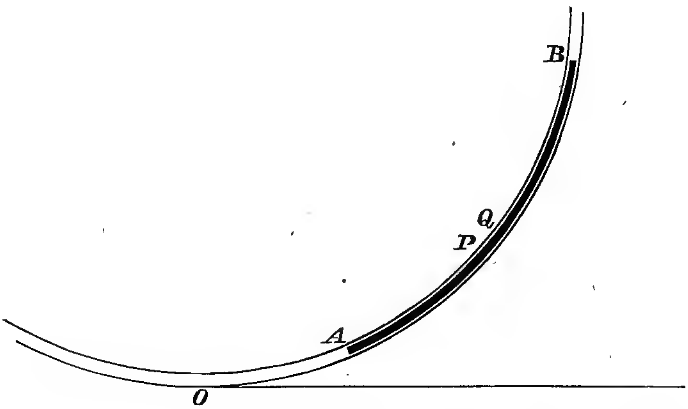
Taking AB as the chain, we let s represent the length OA of the axis of the tube, measured from a fixed point O, and let σ represent the length AP of the chain.

Then if T be the tension at P and $T + \delta T$ at Q, and $R\delta\sigma$ be the pressure of the tube on the element PQ, the equations of motion of the element, the mass of which is $m\delta\sigma$, are, observing that the velocity and the tangential acceleration of every point of the chain are the same,

$$m\delta\sigma\,.\,\ddot{s} = \delta T + m\delta\sigma S,$$

and $$md\sigma\frac{\dot{s}^2}{\rho} = T\frac{\delta\sigma}{\rho} + m\delta\sigma N + R\delta\sigma,$$

where S and N are the tangential and normal forces per unit of mass.



Taking l as the length of chain and integrating the first equation from A to B,

$$l\ddot{s}=\int_0^l S d\sigma,$$

and the second equation gives

$$R=m\frac{\dot{s}^2}{\rho}-\frac{T}{\rho}-mN.$$

As a particular case consider the *motion of a heavy chain inside a smooth circular tube in a vertical plane.*

Measuring from the vertical radius, and from OA, take $COA=\theta$, and $AOP=\phi$.

The equation of motion of the element PQ, or $ma\delta\phi$, is

$$ma\delta\phi\,.\,a\ddot{\theta}=ma\delta\phi\,.\,g\sin(\theta+\phi)+\delta T,$$

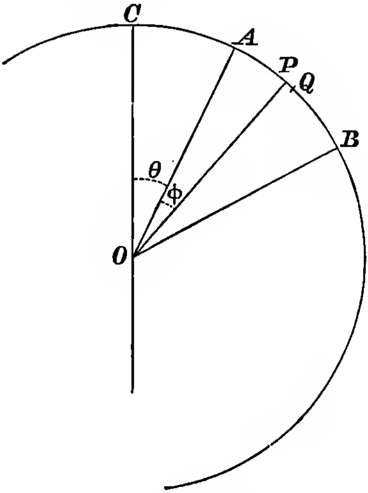
and therefore taking α as the angle subtended by the string,

$$ma^2\phi\ddot{\theta}=T-mga\cos(\theta+\phi)+mga\cos\theta,$$

and $$a\alpha\ddot{\theta} = g\{\cos\theta - \cos(\theta+\alpha)\} = 2g\sin\left(\theta+\frac{\alpha}{2}\right)\sin\frac{\alpha}{2},$$

whence $$a\alpha\dot{\theta}^2 = 2g\sin\frac{\alpha}{2}\left\{\cos\frac{\alpha}{2} - \cos\left(\theta+\frac{\alpha}{2}\right)\right\},$$

if the end A start from C.



For the rate of pressure at any point,

$$ma\delta\phi\,.\,a\dot{\theta}^2 = ma\delta\phi\,.\,g\cos(\theta+\phi) + T\delta\phi - Ra\delta\phi,$$

or $$Ra = mga\cos(\theta+\phi) + T - ma^2\dot{\theta}^2.$$

136. *Motion of a piece of fine chain inside a smooth circular tube which is revolving in its own plane about a point in its circumference.*

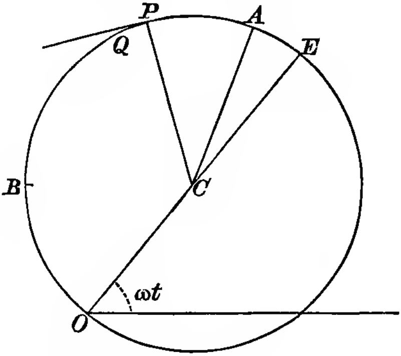
AB being the chain, let $ECA = \theta$, and $ACP = \phi$.

The acceleration of the point P with regard to C, in direction of the tangent is

$$a\frac{d^2}{dt^2}(\theta+\phi), \text{ or } a\theta,$$

and the acceleration of C is $\omega^2 a$ in the direction CO; therefore the equation of motion of the element PQ is

$$ma\delta\phi\,\{a\ddot{\theta}+\omega^2 a\sin(\theta+\phi)\}=\delta T,$$



and, integrating from A to B, that is from $\phi=0$ to $\phi=\alpha$,

$$a\ddot{\theta}=\omega^2\{\cos\theta-\cos(\theta+\alpha)\}=2\omega^2\sin\left(\theta+\frac{\alpha}{2}\right)\sin\frac{\alpha}{2},$$

and therefore $\quad a\dot{\theta}^2=C-4\omega^2\sin\frac{\alpha}{2}\cos\left(\theta+\frac{\alpha}{2}\right),$

the constant being determined by initial conditions.

137. *The equations of motion of a free string in a plane under the action of given forces in the plane.*

If, at any instant, u and v be the tangential and normal velocities of point P of the string, and $\dot{\phi}$ the angular velocity of the tangent, the accelerations of P are

$$\dot{u}-v\dot{\phi},\text{ and }\dot{v}+u\dot{\phi}.$$

Hence the equations of motion of an element PQ are

$$m\delta s\,(\dot{u}-v\dot{\phi})=\delta T+m\delta s\,.\,S,$$

$$m\delta s\,(\dot{v}+u\dot{\phi})=T\frac{\delta s}{\rho}+m\delta s\,.\,N,$$

ρ being the radius of curvature of the string at P and mS, mN the forces per unit of length,

$$\text{or} \qquad m(\dot{u} - v\dot{\phi}) = \frac{dT}{ds} + mS, \quad m(\dot{v} + u\dot{\phi}) = \frac{T}{\rho} + mN.$$

The string being inextensible, the geometrical condition is that the motion of Q relative to P is ultimately perpendicular to PQ, and leads to the equations,

$$(u + \delta u)\cos\delta\psi - (v + \delta v)\sin\delta\psi = u,$$

$$\frac{\delta\phi}{\delta t} = \frac{(v + \delta v)\cos\delta\psi + (u + \delta u)\sin\delta\psi - v}{\delta s},$$

$\delta\psi$ being the angle between the tangents at P and Q; or ultimately

$$\frac{du}{ds} = \frac{v}{\rho}; \text{ and } \frac{d\phi}{dt} = \frac{dv}{ds} + \frac{u}{\rho}.$$

138. A particular case is that of an endless chain originally at rest under the action of conservative forces, and the reactions of smooth curves, and set in motion in such a manner that each point of the chain begins to move in the direction of the tangent at the point.

In this case the chain will retain its form.

For, if v be the velocity of each point of the chain, the equations of motion of an element δs are,

$$0 = m\delta s \,.\, P + \delta T, \quad m\delta s\,\frac{v^2}{\rho} = m\delta s \,.\, Q + \frac{T\delta s}{\rho};$$

$$\text{or} \qquad 0 = m\,P + \frac{d}{ds}(T - mv^2), \quad 0 = mQ + \frac{1}{\rho}(T - mv^2),$$

$$\text{leading to} \qquad P + \frac{d}{ds}(Q\rho) = 0,$$

as the equation giving the form of the curve when there is no motion, and it therefore follows that the chain retains its form, but that the tension at each point is increased by mv^2.

This question is discussed in the solutions of the Tripos Questions for 1854 by Mr Walton and the late Bishop Mackenzie.

Mr W. Froude, in a letter in *Nature*, in November 1875, described the experimental fact that a heavy endless chain, placed over a drum, and made to revolve rapidly, retains unchanged the catenary form which it assumes when not in motion; and further that if an indentation in the form be made by a blow from a heavy hammer, the indentation, if the rotation be very rapid, remains for a considerable time.

In the latter case the tension becomes so great that the action of gravity is unimportant, although of course, the action of gravity will, in time, remove the indentation and restore the catenary form.

139. *The equation for determining the initial tension of a string.*

The problem to be considered is that of a string on the point of motion, under the action of given forces, as for instance a string which being in equilibrium is cut at any point.

Let PQ (δs) be an element of the string, and $m\delta s \,.\, S$, $m\delta s N$ the tangential and normal forces acting upon it.

Take α, β, $\alpha + \delta\alpha$, $\beta + \delta\beta$ as the tangential and normal accelerations at P and Q respectively, and T, $T + \delta T$ as the initial tensions at P and Q.

Then, $\delta\phi$ being the angle between the tangents at P and Q,

$$m\delta s \,.\, \alpha = \delta T + m\delta s \,.\, S, \text{ and } m\delta s \,.\, \beta = (T + \delta T)\sin\delta\phi + m\delta s N,$$

or, ultimately, $$m\alpha = \frac{dT}{ds} + mS, \quad m\beta = \frac{T}{\rho} + mN,$$

where ρ is the radius of curvature.

We have also the geometrical condition that the accelerations of P and Q in the direction PQ are the same; and hence, $\delta\phi$ being the angle between PQ produced and the tangent at Q, we obtain

$$(\alpha + \delta\alpha) \cos \delta\phi - \alpha - (\beta + \delta\beta) \sin \delta\phi = 0,$$

or, ultimately, $$\frac{d\alpha}{ds} - \frac{\beta}{\rho} = 0,$$

and, therefore, from the mechanical equations above,

$$\frac{d^2T}{ds^2} + m\frac{dS}{ds} = \frac{T}{\rho^2} + \frac{mN}{\rho}.$$

If one end of the string be fixed, we have, in determining the constants resulting from integration, to introduce the condition that the tangential acceleration at the fixed point is zero, and, if either end of the string be moveable on a fixed curve, the condition thereby introduced is that the acceleration of that end of the string in the direction of the normal to the curve is zero.

Mathematical Journal, 1864.

For example suppose that a catenary, the upper ends of which are fixed, is severed at the vertex.

We have $s = c \tan \phi$, $S = -g \sin \phi$, $N = -g \cos \phi$,

and therefore $$\frac{dS}{ds} = -g \cos \phi \frac{d\phi}{ds} = \frac{N}{\rho}.$$

The equation is $\frac{d^2T}{ds^2} = \frac{T}{\rho^2}$, or, transforming,

$$\frac{d^2T}{d\phi^2} \cos \phi - 2\frac{dT}{d\phi} \sin \phi - T \cos \phi = 0.$$

Integrating $$\frac{dT}{d\phi} \cos \phi - T \sin \phi = C,$$

and therefore $$T \cos \phi = C\phi + C'.$$

If $\phi = 0$, $T = 0$, and $T \cos \phi = C\phi$.

If α be the value of ϕ at the upper end,

$$\frac{dT}{ds} - mg\sin\phi = 0, \text{ when } \phi = \gamma,$$

which gives
$$C = \frac{mgc\sin\gamma}{\cos\gamma + \gamma\sin\gamma}.$$

For another example take the case of a catenary the ends of which, held apart, can slide on a smooth horizontal rod, and imagine the ends let go.

In this case, as before,

$$T\cos\phi = C\phi + C',$$

but the geometrical conditions are that the tangential acceleration of the vertex is zero, and the vertical acceleration of each end zero.

Therefore $\frac{dT}{ds} = 0$, when $\phi = 0$, which gives $C = 0$, and $T = C'\sec\phi$,

and
$$\alpha\sin\gamma + \beta\cos\gamma = 0, \text{ when } \phi = \gamma,$$

leading to
$$C' = mgc \text{ and } T = mgc\sec\phi.$$

The vertex has no initial acceleration, and the horizontal acceleration of each end is $g\sin\gamma\,(1 - \sin\gamma)$.

140. *A heavy chain, lying on a smooth horizontal plane, receives tangential impulses at one or both ends; it is required to find the impulsive tension at any point, and the direction of the initial motion.*

Taking u and v as the tangential and normal velocities of a point P of the chain, and T as the impulsive tension at P, the equations of motion of an element PQ are

$$m\delta s\,.\,u = \delta T,\ m\delta s\,.\,v = T\frac{\delta s}{\rho};$$

or
$$mu = \frac{dT}{ds},\ \text{and}\ mv = \frac{T}{\rho};$$

and we have besides the equation of continuity expressing the fact that in the limit, the velocity of Q in direction of the tangent at P is the same as the velocity of P in that direction.

This condition gives us

$$(u+\delta u)\cos\delta\phi-(v+\delta v)\sin\delta\phi=u,$$

or

$$\frac{du}{d\phi}=v, \text{ or } \frac{du}{ds}=\frac{v}{\rho}.$$

We hence obtain for the impulsive tension, the equation,

$$\frac{d^2T}{ds^2}=\frac{T}{\rho^2}.$$

If the chain be heterogeneous, that is, if m be a variable quantity, the equation is

$$\frac{d^2T}{ds^2}-\frac{1}{m}\frac{dm}{ds}\frac{dT}{ds}=\frac{T}{\rho^2}.$$

As an example take the case of a piece of chain in the form of a portion of a catenary, bounded by a chord parallel to its directrix, and suppose that equal tangential jerks are applied simultaneously at its two ends.

Since $s=c\tan\phi$, we obtain

$$\cos\phi\frac{d^2T}{d\phi^2}-2\sin\phi\frac{dT}{d\phi}-T\cos\phi=0,$$

and therefore

$$T\cos\phi=C\phi+C'.$$

At the vertex, $u=0$, and at each end, $T=P$, if P be the tangential jerk; and we obtain, γ being the extreme deflection,

$$T\cos\phi=P\cos\gamma.$$

Moreover $mcu=P\cos\gamma\sin\phi$, and $mcv=P\cos\gamma\cos\phi$,

and therefore

$$v=u\cot\phi,$$

shewing that every point of the chain begins to move in the direction parallel to the axis of the catenary.

Ex. (2). *If a chain in the form of a quadrant of a circle, the density at any point of which, measured from one end varies as ϵ^{θ}, receive a tangential jerk at the other end,* we have

$$\frac{1}{m}\frac{dm}{ds}=\frac{1}{a},$$

and our equation is

$$\frac{d^2T}{d\theta^2}-\frac{dT}{d\theta}-T=0;$$

the integral of which is given in Chapter II., and the geometrical conditions are that

$$T=0, \text{ when } \theta=0, \text{ and that } T=P \text{ when } \theta=\frac{\pi}{2};$$

these conditions determine the two constants of integration.

EXAMPLES.

1. A lamina, in the form of a regular polygon, is placed flat on a smooth horizontal plane, and fastened to the plane; a string, the length of which is equal to the perimeter of the polygon, is wound round it, one end being attached to an angular point, and the other end carrying a particle; if the particle be projected horizontally, at right angles to the string, find the time after which the string will be wound up again, and its greatest and least tensions. If the lamina be held in a vertical plane, and the particle be projected in the same plane, with an initial velocity sufficient to keep the string always stretched, find its velocity at the instant the whole string becomes straight, the side of the lamina with which the particle is initially in contact being horizontal and downwards.

2. A small bead is projected with any velocity along a circular wire under the action of a force varying inversely as the fifth power of the distance from a centre of force situated in the circumference. Prove that the pressure on the wire is constant.

3. A particle is placed at the extremity of the vertical minor axis of a smooth ellipse and is just disturbed. Shew that if it quit the ellipse at the end of the latus rectum the eccentricity must satisfy the equation $e^6 + 5e^4 + 3e^2 = 5$.

4. If a particle hanging vertically by a string be projected horizontally, and rise to a point P, and there leave the circular motion, shew that if it recommences circular motion at Q, PQ and the tangent at P to the circle will make equal angles with the vertical.

5. If a particle move under the action of forces F, F', to any number of fixed points, and if q, q',... be the chords of curvature of the path in directions of these forces,

$$2mv^2 = \Sigma\,(Fq).$$

6. A uniform circular ring rotates uniformly in a horizontal plane about its centre. Shew that the greatest possible linear velocity of its particles is independent of the radius of the circle and of the cross section of the ring.

Find the breaking tension in pounds per square inch in the case of a uniform ring of radius a feet which is on the point of breaking when it makes n revolutions a second, the weight of a cubic inch of the material of the ring being that of c ounces.

7. Two equal particles which repel each other with a force $= \mu\,(\text{distance})^{-3}$ are placed on the inner surface of a smooth sphere (rad. a), and their initial distance subtends an $\angle\, 2\alpha$ at the centre: shew that they will perform isochronous oscillations in intervals $= 2\pi a^2 \sqrt{2} \sin \alpha / \sqrt{\mu}$.

8. A particle descends the arc of a smooth cycloid whose axis is vertical from the base to the vertex.

If a horizontal line through the particle meet the circle described on the axis of the cycloid in Q, the velocity of approach of the particle to Q is constant and equal to half the velocity of the particle at the vertex of the cycloid.

9. A particle is placed very near the vertex of a smooth cycloid, $s = 4a \sin \phi$, axis vertical and vertex upwards; find

where the particle runs off the curve, and prove that it falls upon the base of the cycloid at the distance $(\pi/2 + \sqrt{3})\, a$ from the centre of the base.

10. A heavy particle slides, from a cusp, down the arc of a rough cycloid, the axis of which is vertical; prove that its velocity at the vertex will bear to the velocity at the same point when the cycloid is smooth the ratio of

$$\sqrt{\epsilon^{-\mu\pi} - \mu^2} : \sqrt{1 + \mu^2}.$$

11. If a pendulum oscillate in a medium the resistance of which varies as the velocity, prove that the oscillations are isochronous.

12. Two imperfectly elastic particles are constrained to move on a cycloid, whose axis is vertical; they are let fall simultaneously from different points of the cycloid; determine the energy of the particles immediately after the n^{th} collision, and the ultimate loss of energy, as the number of collisions increases without limit.

13. Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t: prove that they will meet at a time

$$2\pi\sqrt{\frac{a}{g}} + \frac{t}{2}$$

after the starting of the first particle, a being the radius of the generating circle.

14. Two equal smooth circles are fixed so as to touch the same horizontal plane, their planes being at different inclinations; two small heavy beads are projected at the same instant along these circles from their lowest points, the velocity of each bead being that due to the height of the highest point of the other circle above the horizontal plane; shew that during the motion the two beads will always be at equal heights above the horizontal plane.

15. A particle is attached to a point in a rough plane inclined to the horizon at an angle α, originally the string is its natural length; prove that the particle will not oscillate unless $\tan\alpha > 3\mu$, where μ is the coefficient of friction.

Between what limits must μ lie in order that the particle may come to rest at the end of its m^{th} semi-oscillation?

16. A fine parabolic groove has its axis vertical and vertex downwards, an elastic string has one extremity attached to the focus, and the other to a particle in the groove; the natural length of the string equals one-fourth the latus rectum, and the weight of the particle is such as to stretch the string to twice its natural length; determine the position of equilibrium, and shew that the time of a small oscillation about it is $2\pi\sqrt{2a/g}$.

17. If a particle, mass m, be acted upon by equal constant forces mf in the directions of the tangent and normal to its path, and if the resistance be $mf\,v^2/k^2$, prove that the intrinsic equation of the path is

$$k^2\left(\epsilon^{\frac{2fs}{k^2}}-1\right)=u^2\left(\epsilon^{2\phi}-1\right),$$

u being the velocity of projection.

18. Two elastic strings the natural length of each of which is $\frac{1}{2}\pi a$, are fastened at a point P in a circlar tube (radius a) of small bore; the strings are stretched in opposite directions, and their other extremities fastened to a particle of given weight. If the plane of the tube be horizontal, and the particle be displaced from its position of equilibrium through an angle less than $\pi/2$, shew that the time of an oscillation is independent of the extent of the displacement.

19. A heavy particle is projected upwards from the vertex, within a smooth parabola whose axis is horizontal, with the velocity due to a fall down the latus rectum ($4a$). Investigate the subsequent motion, and shew that the particle impinges upon the parabola again, at a distance $3a\sqrt{13}$ from the vertex, with a velocity that bears to the velocity of projection the ratio $\sqrt{5}:\sqrt{2}$.

20. A particle describes a circular arc under the action of a constant force not tending to the centre; shew that it will oscillate through a quadrant.

21. A particle is moving on the convex side of a rough equiangular spiral towards the pole, under the action of a force to the pole $=\kappa v$, where v is the velocity at distance r. If V be the velocity at distance a, and t the time of moving from distance r to distance a, shew that, μ being the coefficient of friction,

$$r^{1+\mu\tan\alpha}-a^{1+\mu\tan\alpha}=\frac{a^{\mu\tan\alpha}V}{\kappa}\,\frac{\cos\alpha+\mu\sin\alpha}{\cos\alpha-\mu\sin\alpha}\left\{1-\epsilon^{(-\cos\alpha+\mu\sin\alpha)\kappa t}\right\},$$

where α is the angle of the spiral.

22. Having given that the normal acceleration varies as the square of the velocity parallel to the axis of x, find the path; and prove that if this path be described under the action of a force parallel to the axis of y, then the whole acceleration at any point is proportional to the velocity.

23. If the curve whose intrinsic equation, referred to a horizontal tangent, is $s^2\sin\phi=a^2$ be described by a particle under the action of gravity, find the time of descent from any point of the curve to the horizontal tangent.

24. A particle, mass m, moves in a smooth circular tube of radius a, under the action of a force, μm (distance), to a point inside the circle at a distance c from its centre; if the particle be placed very nearly at its greatest distance from the centre of force, prove that it will pass over the quadrant ending at its least distance in the time

$$\sqrt{a}\log(\sqrt{2}+1)/\sqrt{\mu c}.$$

25. A particle P attached by a string of length a to a fixed point C describes a circle in one plane, under the action of a uniform repulsive force f emanating from a fixed point O; OC being $=c$ and $>a$. If V be the velocity of the particle when at its greatest distance from O and v its velocity after describing an $\angle\,\theta$ from that position, shew that

$$v^2=V^2-2f\{c+a-\sqrt{c^2+a^2+2ac\cos\theta}\}.$$

Find also the tension of the string and shew that

$$V^2\nless 5fa.$$

26. A heavy particle is projected in a resisting medium; if v be the velocity at any time, ϕ the inclination to the vertical of the direction of motion, and f the retardation, prove that

$$\frac{1}{v}\frac{dv}{d\phi}+\cot\phi+\frac{f}{g\sin\phi}=0.$$

If $f=\mu v^3$, find v in terms of ϕ.

27. Prove that the curve possessing the property that the product of the distances of any point on it from two fixed points is constant, may be described with uniform velocity under the action of two forces, each tending to one of the fixed points, and varying as the distance from the other, the absolute intensities of the forces being the same.

28. Prove that a particle can describe a parabola under a repulsive force in the focus varying as the distance and another force parallel to the axis always of three times the magnitude of the former; and that if two equal particles describe the same parabola under the action of these forces, their directions of motion will always intersect on a fixed confocal parabola.

29. Find the time of a small oscillation of a particle suspended from a point by a string of length l, when the square of α, the angle of oscillation, is neglected; and shew that the time will be $\pi\left(1+\frac{\alpha^2}{16}\right)\sqrt{\frac{l}{g}}$, if the approximation include the square of α.

A weight is drawn up uniformly and slowly with velocity u by means of a crane; shew that the times of small oscillations will decrease at first in arithmetical progression, the common difference being $\pi^2u/2g$.

30. Two equal particles, connected by a fine string, are placed in a circular tube, to one point of which they are attracted with a force varying inversely as the distance; one of the particles being initially at its greatest distance from the centre of force, and v, v' being the velocities with which

they successively pass through the point whose distance from the centre of force is 90°, shew that $\epsilon^{-\frac{v^2}{\mu}} + \epsilon^{-\frac{v'^2}{\mu}} = 1$.

31. Two equal particles are connected by a string passing through a small hole in a smooth horizontal table, one particle hanging down, and the other held on the table; if the latter be projected along the table at right angles to the string with the velocity $\sqrt{2gc}$, prove that the initial radius of curvature of its path is $4c$.

32. Within a smooth circular tube of radius a held fixed in a vertical plane lies a light string of length greater than half the circumference of the tube. The string carries equal weights at its ends, which balance within the tube, and the string subtends at the centre an angle $2(\pi - \alpha)$. If they be slightly disturbed, shew that the time of a small oscillation is the same as that of a simple pendulum of length $a \sec \alpha$.

33. A particle moves under the action of a central force which is such that the normal acceleration on the particle is constant: find a differential equation of the first order to the path of the particle, and shew that $r^3 \sin 3\theta = a^3$ is a particular integral.

34. A heavy particle moves on a smooth curve in a vertical plane, the form of the curve being such that the pressure on the curve is always m times the weight of the particle: prove that the time of a complete revolution is $2\pi m\sqrt{a}/\sqrt{g(m^2-1)^3}$, and that the length of the vertical axis of the curve is $2ma/(m^2-1)^2$, the whole length of the curve being $\pi a(2m^2+1)/(m^2-1)^{\frac{5}{2}}$.

35. A small smooth heavy bead runs on a string fastened at two points in the same vertical line: the string is originally vertical and the bead in its lowest possible position, the bead is then projected so that it proceeds to describe a portion of an ellipse, the string being at first tight; prove that if the string becomes slack when the bead is at the extremity of one of the equi-conjugate diameters of the

ellipse, then the free path of the bead will pass through the other extremity of the same diameter, and the latus rectum of its free path will be to that of the ellipse as $1 : 2\sqrt{2}$.

36. On a wire in the form of a parabola with axis vertical and vertex downwards is a bead attached to the focus by an elastic string whose natural length is one eighth of the latus rectum and whose modulus is equal to the weight of the bead. Prove that the time of a small oscillation is $2\pi\sqrt{a/g}$, where $4a$ is the length of the latus rectum.

37. A particle slides down the arc of a vertical parabola with vertex downwards starting from rest at a height h above a horizontal line through the vertex. Shew that the time of descent to the vertex is equal to

$$\sqrt{\left\{\frac{2(a+h)}{g}\right\}}\, E^1\left\{\sqrt{\left(\frac{h}{a+h}\right)}\right\},$$

where $E^1(k)$ denotes the complete second elliptic integral to modulus k.

38. A circle, centre C, is described round an internal centre S of attractive force; shew that the force varies as

$$\frac{r}{(a^2+r^2-c^2)^3},$$

where $SP=r$, $a=$ radius of circle, $SC=c$.

If there be two equal centres of force S, S' such that SCS' is a straight line, and

$$SC=S'C=c,$$

then taking each force to be $\dfrac{\mu r}{(a^2+r^2-c^2)^3}$, shew that a particle will describe a circle, centre C and radius a $(>c)$, if projected from a point in the line CS distant a from C with a velocity $\dfrac{\sqrt{\mu}\,(a^2+c^2)}{2a(a^2-c^2)}$ perdendicular to CS.

39. A smooth wire is bent into the form of a circle radius a, and rotates with uniform angular velocity ω about a

vertical axis through the centre which makes an angle α with the plane of the circle. If a smooth bead slide on the wire, shew that the equation of motion of the bead along the wire is

$$\frac{d^2s}{dt^2} = a\omega^2 \cos^2\alpha \cos\frac{s}{a} \sin\frac{s}{a} - g\cos\alpha \sin\frac{s}{a},$$

where s is measured from the lowest point. Hence find the position of equilibrium of the bead, and the time of a small oscillation about that position.

40. A heavy particle hanging by a string from a fixed point O is projected horizontally and describes a portion of a circle greater than a quadrant until when it arrives at a point P the string slackens and it begins to move in a parabola: shew that the circle is the circle of curvature of the parabola at P, and that if OP be produced to meet the directrix the locus of the intersection is a circle concentric with the given one.

41. A heavy bead slides on a smooth fixed vertical circular wire of radius a: if it be projected from the lowest point with a velocity just sufficient to carry it to the highest, prove that the radius through the bead will, in a time t, turn through an angle

$$2\tan^{-1}\left(\sinh\sqrt{\frac{g}{a}}\,t\right).$$

42. If a particle move on an ellipse under a force to the centre $= mr - nr\log\frac{c-r}{c+r}$, where $c^2 = a^2 + b^2$, and N be the pressure on the curve, ρ the radius of curvature, prove

$$N\rho = 2ncr + \text{constant}.$$

If the velocity vanish at an extremity of the major axis and $\frac{m}{n} = \log\frac{c+a}{c-a}$, then

$$N\rho = 2nc(r-a).$$

43. Snow is uniformly spread over the surfaces of a conical pinnacle and of the hemispherical dome of a building.

It begins to slide off, starting at the highest point and clearing a path as it goes. Prove that the motion in the two cases is the same as that of a free particle moving on the surfaces under the action of a vertical acceleration equal to one-fifth and one-third the acceleration of gravity respectively.

44. If a heavy particle be projected with a given velocity and in a given direction prove that the initial curvature of the path is independent of the resistance. Shew that, if the resistance $= \mu$ (velocity)^2, the radius of curvature (ρ), at the point where the tangent is inclined at an angle ϕ to the vertical, satisfies the equation,

$$\sin\phi \frac{d\rho}{d\phi} + 3\rho\cos\phi + 2\mu\rho^2\sin\phi = 0\,;$$

and that, if $1/\rho'$ be the curvature at the other point where the inclination to the vertical is the same, the curvature at the vertex is

$$\frac{1}{2\sin^3\phi}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right).$$

45. A particle is projected along a rough tube, and is acted on by a force always bisecting the angle between the tangent and the normal, and varying directly as the curvature and as the sine of the angle (ψ) the tangent makes with a fixed straight line. Shew that the velocity depends only on ψ, whatever be the shape of the tube; and if it is zero when $\psi = \epsilon$, where the coefficient of friction is $\frac{1}{2}\cot\epsilon$, shew that the time of describing an arc s varies as

$$\int_0^s \frac{ds}{\sqrt{\sin(\psi - \epsilon)}}.$$

46. A tube of uniform bore in the form of an equiangular spiral is revolving uniformly with angular velocity ω in a horizontal plane about a vertical axis through its pole, and within the tube is a smooth uniform chain of length $2l$ and mass m, which is initially at rest with its middle point at a given distance from the pole; find the space described by the chain along the tube in a given time, and shew that

the tension at any point of the chain is $m\omega^2 \cos^2 \alpha \, (l^2 - x^2)/4l$, where x is the arcual distance of the point from the middle point, and α is the angle of the spiral.

47. A string of varying density slides in a smooth cycloidal tube whose axis is vertical and vertex downwards. Shew that if the string be let fall from any position in which its whole length is within the tube, its centre of gravity will reach the vertex in the same time.

48. The bob of a simple cycloidal pendulum starts from rest at the cusp. If the string break in any subsequent position after passing the axis of the curve, the locus of all possible positions of the focus of the trajectory subsequently described by the bob is given by the equation

$$\frac{x}{l} = \pi - \cos^{-1}\left(1 - \sqrt{\frac{y}{l}}\right) + \left(1 + \sqrt{\frac{y}{l}}\right)\sqrt{2\sqrt{\frac{y}{l}} - \frac{y}{l}},$$

the middle point of the base of the cycloid being taken as origin and the length of the string being $4l$.

49. A string of infinite length is laid on a smooth table in the form of a portion of one branch of the curve $r^n \sin n\theta = a^n$, so that one extremity of the string is at a finite distance from the origin of polar coordinates; to this end a tangential impulse is applied, so that the initial direction of motion of each point of the string and the radius vector to the point are equally inclined to the corresponding tangent. Shew that the impulsive tension at any point $\propto r^{-(n-1)}$ and the density of the string must

$$\propto r^{1-2n} \left(r^{2n} - a^{2n}\right)^{\frac{2n-1}{2n}}.$$

50. A uniform string falls freely in one plane under the influence of gravity: prove that the angular acceleration of the tangent at any point is

$$\frac{2}{m\sqrt{\rho}} \frac{d}{ds}\left(\frac{T}{\sqrt{\rho}}\right),$$

where T is the tension at the point, ρ the radius of curvature, s the arc measured from a fixed point of the string, and m the mass of a unit of length.

51. The equation to a curve is

$$(u-a)\{(u-a)^2 - b^2(\theta+\alpha)^{\frac{14}{3}}\} = 0;$$

A particle m, placed on the curve, and acted upon by the force to the origin

$$mh^2u^2\{u + \tfrac{28}{9}b^{\frac{6}{7}}(u-a)^{\frac{1}{7}}\},$$

is projected tangentially so that its velocity, when it arrives at the distance a^{-1} from the pole, shall be ha; prove that its pressure on the curve will be always zero.

52. A bead can slide on a smooth circular arc AB and is attracted by it, the force to any point being $f(r)$: if it be displaced from its position of equilibrium, the time of oscillation will be $2\pi/\sqrt{2\cos\alpha f(AC)}$, where C is the middle point of AB, and 2α the angle AC subtends at the centre of the circle.

53. Prove that a lemniscate can be described freely by a particle under the action of two centres of force of equal intensity in the foci, each varying inversely as the distance, and that the velocity will always be equal to $\sqrt{4\mu/3}$, μ/r being the acceleration of either force on the particle at a distance r.

54. A heavy particle, mass m, falls down a smooth cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is $mv^2/2c$, and the distance of the starting point from the vertex is c; prove that the time to the cusp is $\sqrt{8a(4a-c)}/\sqrt{gc}$, $2a$ being the length of the axis.

55. A particle is acted on by two forces, tending to the foci of an ellipse whose major axis is $(2a)$, and varying according to the law $\mu(r^3+8a^3)/8a^3r^2$, the absolute intensities being the same. Shew that, if it be projected along the tangent to the ellipse with a certain velocity, then it will continue to describe the ellipse freely, and its velocity, in any position given by the focal distances (r, r'), will be

$$n(r^2+rr'+r'^2)/2\sqrt{rr'},$$

(n) being the mean motion in the ellipse under a force μ/r^2 to a focus.

56. In the case of a piece of uniform chain, on a smooth horizontal table, receiving a tangential jerk at one end;

(1) Find the form of the curve if the initial direction of motion of any point makes a constant angle with the tangent at that point.

(2) Find the impulsive tension at any point when the form is such that the line-mass at any point varies as the curvature.

(3) Prove that if the impulsive tensions follow the same law as the tensions in a non-uniform string hanging in the same curve under the action of gravity, then

$$2\,.\,\frac{d^2\rho}{ds^2}=\left.\overline{\frac{d\rho}{ds}}\right|^2+4.$$

(4) If all the particles of the chain start with equal velocities, prove that the form must be that of a straight line or of a catenary.

57. AB is a diameter of a horizontal circular wire, and a particle, free to move on the wire, is repelled from A by a constant force equal to twice its weight. Shew that if placed at an angular distance 2α from B, it will oscillate about B in the same time as if it were oscillating under the action of gravity alone on the same circular wire placed with its plane vertical through an angular distance α on each side of the lowest point.

58. If a projectile move in a medium the resistance of which varies as the square of the velocity, and if ρ, ρ' be the radii of curvature of the path at two points at each of which the direction of motion is inclined at an angle ϕ to the horizon, and r the radius of curvature at the highest point: shew that

$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{2\cos^3\phi}{r}.$$

59. Two points A, B, such that AB is inclined to the horizon at an angle $\left(\frac{\alpha}{2}\right)$, are connected by a wire made up

of two equal circular arcs of radius (a), placed so that the tangents at the extremities A, B are horizontal, while the other ends meet at a point C in AB, so that the two arcs have at C a common tangent. If a ring be projected from A, with velocity u, it will come to rest at B, provided

$$\frac{u^2}{4ga} = \frac{(2\mu^2 - 1)(e^{2\mu\alpha}\cos\alpha - 1) + 3\mu e^{2\mu\alpha}\sin\alpha}{4\mu^2 + 1},$$

μ being the coefficient of friction.

60. A chain, in equilibrium in one plane, receives a tangential impulse at a given point; prove that the impulse at any point is given by the equation

$$\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{dT}{d\phi}\right) = \frac{T}{\rho},$$

ρ being the radius of curvature, and ϕ the inclination of the tangent to a fixed line.

A uniform chain hangs in equilibrium over two smooth pegs in the same horizontal; if equal vertical impulses be applied simultaneously to the two free ends, find the impulsive tension at any point, and prove that the initial velocity of the vertex of the catenary is to the velocity which would be imparted to each of the straight pieces of chain, if disjointed from the catenary, as $1 : 1 + \sin\alpha$, where α is the greatest inclination of the catenary to the horizontal plane.

61. A particle is attracted to two centres of force S and H by forces which each follow the law

$$r\left\{\mu + \frac{\mu'}{(r^2 - b^2)^3}\right\},$$

where μ and μ' are the same for both centres of force. Shew that the particle can describe a circle whose centre is midway between S and H, if b be the length of the tangent drawn from either centre of force to the circle.

62. A shot is fired in an atmosphere in which the resistance varies as the cube of the velocity. If f be the retardation when the shot is ascending at an inclination α to

the horizon, f_0 when it is moving horizontally, and f' when it is descending at an inclination α to the horizon; then

$$\frac{1}{f'}+\frac{1}{f}=\frac{2\cos^3\alpha}{f_0}, \text{ and } \frac{1}{f'}-\frac{1}{f}=\frac{2\sin\alpha\,(3-2\sin^2\alpha)}{g}.$$

63. A portion of a heavy uniform string is placed on the arc of a four-cusped hypocycloid, so as to occupy the space between two cusps, the tangents at which are horizontal and vertical respectively, and runs off the curve at the lower cusp; prove that the velocity which the string will have when the whole of it has just left the curve will be the velocity due to nine-tenths the length of the string.

64. A fine chain of given length is contained in a smooth circular tube which rotates uniformly in a horizontal plane about a fixed point in the circumference; if the chain subtend an angle 2α at the centre, and if one end be initially fastened to the tube at the end of the diameter through the fixed point, and be released, prove that the angular motion of the chain relative to the tube will be given by the equation

$$\alpha\dot{\theta}^2 = 2\omega^2\sin\alpha\,(\cos\theta-\cos\alpha).$$

65. A piece of string in the form of part of the curve, $r = a\epsilon^{\theta\cot\alpha}$, the density at any point of which varies as $r^{-\tan^2\alpha}$ lies on a smooth horizontal plane, being bounded by $\theta = 0$, and $\theta = \beta$. If a tangential jerk be applied at the end $\theta = 0$, find the tangential impulse at any point, and prove that the initial direction of motion of every point makes an angle with the normal equal to the angle of the spiral.

66. A string is in equilibrium in the form of a circle under the action of a central repulsive force; if the string be cut at any point, prove that the tension at a point, the angular distance of which from the point of section is θ, is instantaneously changed in the ratio

$$\left(\epsilon^{\frac{\theta}{2}}+\epsilon^{-\frac{\theta}{2}}\right)^2 : 2.$$

67. A heavy string, passing under and in contact with the arc of a fixed vertical circle, centre O, has its ends fastened to two points P, Q on the circle, each at an angular distance α from the lowest point, the pressure at which is zero.

If the circle be suddenly removed prove that the tension at the lowest point is changed at once in the ratio

$$\sin\alpha : \sinh\alpha.$$

Examine the case in which $\alpha = \pi$.

Also determine the new tension when the ends are allowed to be moveable on smooth rods OP, OQ at the instant the circle is removed.

Further, if the ends are allowed to be moveable on smooth rods BP, BP' from the highest point B of the circle, the tension at the lowest point is changed in the ratio

$$\sin\frac{\alpha}{2} : \sinh\alpha\cos\frac{\alpha}{2} - \cosh\alpha\sin\frac{\alpha}{2}.$$

CHAPTER IX.

MOTION IN THREE DIMENSIONS.

141. THE fundamental equation of Kinetics being that the time-flux of the momentum of a particle, in any assigned direction, is equal to the sum of the acting forces in that direction, and the resulting equations for the motion of a particle in three dimensions being given, in different forms in Art. (52), we proceed to employ these equations in some particular cases.

Motion of a heavy particle in contact with fixed smooth curves or surfaces.

Measuring z vertically downwards, and taking the acceleration along the tangent to the path of the particle, we obtain

$$mv\frac{dv}{ds} = mg\frac{dz}{ds},$$

and therefore $\frac{1}{2}m(v^2 - u^2) = mg(z - c),$

if the particle start with the velocity u from the level c.

This is in effect the equation of energy, but it must be carefully borne in mind that, in this case, the system consists of a particle and the earth, and that we are neglecting the kinetic energy acquired by the earth in consequence of the attraction between it and the particle.

142. *Motion of a heavy bead sliding down a smooth wire in the form of a helix with its axis vertical.*

If α be the inclination of the arc of the helix to its base, R and R' the reactions in directions of the principal normal and binormal, the equations of motion are

$$mv\frac{dv}{ds} = mg\sin\alpha,\ \frac{mv^2}{a\sec^2\alpha} = R,\ 0 = R' - mg\cos\alpha.$$

From these equations, $v^2 = 2gz$, if z be the height through which the bead has fallen, and the resulting reaction is equal to $\sqrt{R^2 + R'^2}$.

143. In general, if a particle move in free space, or in contact with smooth curves or surfaces, the equation of motion, obtained by taking the forces in direction of the tangent, is

$$mv\frac{dv}{ds} = mS,$$

mS being the resultant of the acting forces in direction of the tangent.

If the particle have a velocity u at the point P and a velocity v at the point Q,

$$\tfrac{1}{2}m(v^2 - u^2) = \int mSds,$$

the integral being taken from P to Q.

Now, in accordance with the definition of potential energy in Art. 47, the change of potential energy of the system, consisting of a particle in a field of force, that is, of a particle, and attracting masses the kinetic energy of which may be neglected, is the work which would have to be done against the forces of the system in order to move the particle from P to Q, and therefore, if U be the potential when the particle is at P, and V when it is at Q,

$$V - U = \int - mSds,$$

the integral being taken from P to Q.

We hence obtain

$$\tfrac{1}{2} m (v^2 - u^2) = U - V$$

or

$$\tfrac{1}{2} mv^2 + V = \tfrac{1}{2} mu^2 + U,$$

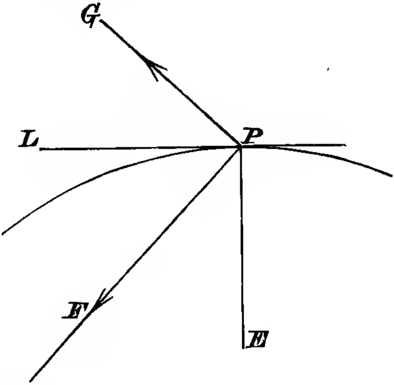
which is the equation of energy.

The total energy being constant, we notice that the force on the particle, which is the time-flux of the momentum, is the negative space-flux of the potential energy.

In other words the force in any direction, at any point of the field, is the rate of exhaustion, in that direction, per unit of linear space, of the potential energy.

144. *Reaction of a smooth surface on a particle which is moving in contact with the surface.*

Let the figure be a section of the surface by a plane perpendicular to the line of motion of the particle P, and let PF be the principal normal of the path, and PG the binormal.



The accelerations in these directions being v^2/ρ', and zero, where ρ' is the radius of absolute curvature of the path, it follows that the resulting acceleration in direction of the normal PE to the surface is $v^2 \cos \phi/\rho'$, if ϕ be the angle FPE or, by Meunier's theorem, v^2/ρ, where ρ is the radius of curvature of the normal section of the surface by the plane through the tangent to the path.

Hence, if R be the pressure, measured inwards, and N the acting force in the direction of the normal to the surface,

$$mv^2/\rho = N + R.$$

If U be the acting force in direction of the tangent PL to the surface in the plane EPF,

$$mv^2/\rho' . \sin\phi = U,$$

and therefore $$mv^2 \tan\phi = \rho U,$$

an equation which determines the position of the osculating plane of the path.

145. *If no forces are in action the path is a geodesic.*

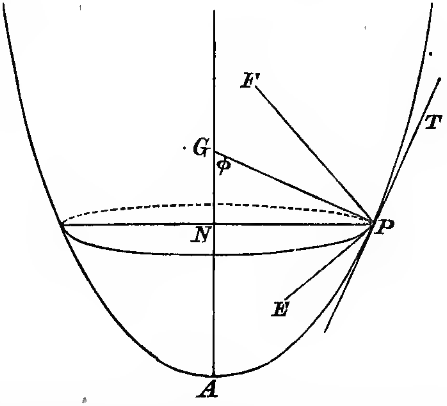
For, taking the equation of motion in direction of the binormal,

$$0 = R\sin\phi;$$

therefore $\phi = 0$, or the osculating plane is a normal plane.

This result is equally true if the surface be a rough surface, for the same equation exists.

146. *Motion of a heavy particle in a smooth surface of revolution the axis of which is vertical.*



Measuring z upwards, and employing cylindrical co-ordinates, the accelerations are

$$\ddot{r} - r\dot{\theta}^2,\ \ 2\dot{r}\dot{\theta} + r\ddot{\theta},\ \text{ and } \ddot{z},$$

and therefore, taking the acceleration in direction of the tangent PT to the meridian,

$$\ddot{z} \sin \phi + (\ddot{r} - r\dot{\theta}^2) \cos \phi = -g \sin \phi.$$

Also, there being no horizontal force perpendicular to the plane APN,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0,$$

and therefore $$r^2\dot{\theta} = h.$$

The equation of the surface, $z = f(r)$, being given, these equations determine the motion.

Taking the acceleration in direction of the normal PG, we have, for the pressure,

$$m \{\ddot{z} \cos \phi - (\ddot{r} - r\dot{\theta}^2) \sin \phi\} = R - mg \cos \phi.$$

Observing that $\tan \phi = dz/dr$, and that

$$\dot{r} = \frac{h}{r^2} \frac{dr}{d\theta}, \text{ and } \dot{z} = f'(r) \frac{h}{r^2} \frac{dr}{d\theta},$$

the first equation becomes

$$\frac{1}{r^4}[1 + \{f'(r)\}^2] \frac{d^2r}{d\theta^2} + \left[\frac{f'(r) f''(r)}{r^4} - \frac{2}{r^5}[1 + \{f'(r)\}^2]\right] \left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^3} = -\frac{g}{h^2} f'(r), \dots\dots\dots (1),$$

which is the differential equation of the projection of the path on a horizontal plane.

Multiplying by $2\dfrac{dr}{d\theta}$ and integrating, we obtain

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 [1 + \{f'(r)\}^2] + \frac{1}{r^2} = C - \frac{2g}{h^2} f(r) \dots\dots (2).$$

If the path be a horizontal circle of radius a, we at once obtain from the equation (1)

$$h^2 = ga^3 f'(a).$$

Or, we can obtain this result by considering that the acceleration to the centre, v^2/a, is maintained by the action of gravity and the pressure of the surface.

If we employ the principles of energy and momentum, leading to the equations,

$$\tfrac{1}{2} m \{\dot{z}^2 + \dot{r}^2 + r^2\dot{\theta}^2\} = C - mgz, \text{ and } r^2\dot{\theta} = h,$$

we arrive at once at the equation (2).

147. *To find the apsidal angle of the projection on a horizontal plane of the path when it is nearly circular.*

When the particle is moving in a circle, we can imagine a slight disturbance of the motion, as for instance by the action of a small impulse in the vertical plane through the axis and the particle. The value of h will then be unchanged, and if we assume $r = a + v$, where v is a very small quantity, and neglect $\left(\frac{dv}{d\theta}\right)^2$, the differential equation becomes

$$[1 + \{f'(a)\}^2] \frac{d^2v}{d\theta^2} - a - v + \frac{1}{a^3 f'(a)} (a^4 + 4a^3 v) \{f'(a) + f''(a)\, v\} = 0,$$

or $$\{1 + (f'a)^2\} \frac{d^2v}{d\theta^2} + \left\{3 + \frac{af''(a)}{f'(a)}\right\} v = 0,$$

so that the apsidal angle is

$$\pi \sqrt{1 + \{f'(a)\}^2} \,/\, \sqrt{3 + af''(a)/f'(a)}.$$

This shews that the motion is stable, provided

$$3 + af''(a)/f'(a)$$

is positive.

148. *A heavy particle is projected horizontally along the inside of a surface of revolution; it is required to find the initial curvature of its path.*

PG being the normal to the surface at P, let PE and PF, in the figure of Art. (146), be the directions of the principal normal and of the binormal of the path.

If ρ' be the radius of absolute curvature, the acceleration in the direction PE is v^2/ρ', and therefore, if $EPG = \psi$, the acceleration in the direction PT is $v^2 \sin\psi/\rho'$,

and therefore $$v^2 \sin\psi/\rho' = g \sin\phi.$$

PG is the radius of curvature of the normal section of the surface perpendicular to the plane ANP, and therefore, by Meunier's theorem,

$$\rho' = PG \cos\psi = r\cos\psi / \sin\phi, \text{ if } r = PN.$$

Hence $$v^2 \tan\psi = gr,$$

and $$\rho' = \frac{v^2 r}{\sin\phi \sqrt{v^4 + g^2 r^2}}.$$

149. *Motion of a heavy particle on the surface of a sphere.*

In the figure of Art. (146) let G be the centre of the sphere and take c for the radius.

The equation, $r^2\dot{\theta} = h$, becomes $c^2 \sin^2\phi \, . \, \dot{\theta} = h$, and the equation of energy is

$$c^2\dot{\phi}^2 + c^2 \sin^2\phi\dot{\theta}^2 = C + 2gc\cos\phi,$$

or $$c^2\dot{\phi}^2 + \frac{h^2}{c^2 \sin^2\phi} = C + 2gc\cos\phi.$$

If the particle be projected horizontally, from the position $\phi = \alpha$, with the velocity v,

$$h = vc\sin\alpha, \quad \dot{\phi} = 0, \quad \text{and} \quad v^2 = C + 2gc\cos\alpha,$$

so that

$$c^2\dot{\phi}^2 + v^2\left(\frac{\sin^2\alpha}{\sin^2\phi} - 1\right) = 2gc(\cos\phi - \cos\alpha).$$

To find the greatest and least altitudes of the particle, put $\dot{\phi} = 0$, then we obtain $\phi = \alpha$, or

$$v^2(\cos\alpha + \cos\phi) = 2gc(1 - \cos^2\phi).$$

It is easily seen that this equation gives one value for $\cos\phi$ lying between $+1$ and -1, and therefore it follows that the whole motion of the particle is comprised between two horizontal planes.

Taking the acceleration in the direction of the normal GP, we find for the pressure, measuring z downwards,

$$m\ddot{z}\cos\phi + m(\ddot{r} - r\dot{\theta}^2)\sin\phi = mg\cos\phi - R,$$

leading to

$$R = mg\cos\phi + mc\dot{\phi}^2 + m\frac{v^2}{c}\,\frac{\sin^2\alpha}{\sin^2\phi},$$

or $$R = mg\,(3\cos\phi - 2\cos\alpha) + mv^2/c.$$

This is in accordance with the general result of Art. (144), for, in this case, the square of the velocity

$$= v^2 - 2gc\,(\cos\alpha - \cos\phi).$$

150. *Motion of a heavy particle on the surface of a smooth cone, the vertex being downwards, and the axis vertical.*

Employing cylindrical co-ordinates, and taking the acceleration along the generating line, we obtain

$$(\ddot{r} - r\dot{\theta}^2)\sin\alpha + \ddot{z}\cos\alpha = -g\cos\alpha,$$

or $$\ddot{r} - r\sin^2\alpha\dot{\theta}^2 = -g\sin\alpha\cos\alpha.$$

Also $r^2\dot{\theta} = h$, and putting u for $\frac{1}{r}$, we find for the differential equation of the projection of the path on a horizontal plane,

$$\frac{d^2u}{d\theta^2} + u\sin^2\alpha = \frac{g\sin\alpha\cos\alpha}{h^2u^2}.$$

151. *Motion of a particle in a smooth plane tube which revolves about an axis in its plane.*

Take the axis of rotation as the axis of z, and ϕ as the inclination of the normal to the axis of z.

Taking S, N, and T as the acting forces in directions of the tangent and normal to the curve and perpendicular to its plane, and R, R' as the reactions in the two last-named directions, the equations of motion are

$$m\,(\ddot{r} - r\dot{\theta}^2)\cos\phi + m\ddot{z}\sin\phi = S,$$

$$m\ddot{z}\cos\phi - m\,(\ddot{r} - r\dot{\theta}^2)\sin\phi = N + R,$$

$$m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = T + R'.$$

Take for instance the case of a parabolic tube with its vertex downwards, revolving uniformly about its axis which is vertical.

We have then

$$S = -mg \sin\phi,\ N = -mg\cos\phi,\ T = 0,$$

$$\dot{\theta} = \omega,\ 4az = r^2, \text{ and } 2a\tan\phi = r,$$

and the first equation becomes

$$\ddot{r}(4a^2 + r^2) + r\dot{r}^2 = (4a^2\omega^2 - 2ag)\,r.$$

Integrating and supposing that initially $r = c$, and $\dot{r} = 0$,

$$r^2(4a^2 + r^2) = 2a(2a\omega^2 - g)(r^2 - c^2).$$

If $2a\omega^2 = g$, the particle will remain, in relative rest, at whatever point it is placed initially; and according as $2a\omega^2 >$ or $< g$, the particle will ascend or descend.

152. *Motion of a heavy bead on a smooth wire in the form of a helix, having its axis vertical, and revolving uniformly about its axis.*

Measuring θ from the foot of the helix, the cylindrical co-ordinates of the bead are

$$a,\ \theta + \omega t,\ z, \text{ where } z = a\theta\tan\alpha.$$

Taking the acceleration along the tangent to the helix,

$$a\theta\cos\alpha + \ddot{z}\sin\alpha = -g\sin\alpha,$$

or $$\ddot{z} = -g\sin^2\alpha, \text{ and } \therefore \dot{z}^2 = 2g\sin^2\alpha\,(h - z).$$

This gives the vertical velocity, and the horizontal velocity

$$= a(\dot{\theta} + \omega) = a\omega + \dot{z}\cot\alpha.$$

For the motion on the arc of the helix,

$$s = z \operatorname{cosec}\alpha \text{ and } \therefore \dot{s}^2 = 2g(h - z).$$

If R, R' be the reactions of the wire on the bead in directions of the principal normal and binormal of the helix, we have,

$$ma(\dot{\theta} + \omega)^2 = R,$$

and $$m(\ddot{z}\cos\alpha - a\ddot{\theta}\sin\alpha) = R' - mg\cos\alpha.$$

153. The motion of a bead on a tortuous wire revolving about a fixed axis, and acted upon by any given forces, can be similarly treated by the use of cylindrical co-ordinates.

Or we can take axes of x and y revolving with the curve, in which case the expression for the acceleration along the tangent to the curve will be

$$(\ddot{x} - \omega^2 x - 2\dot{y}\omega)\frac{dx}{ds} + (\ddot{y} - \omega^2 y + 2\dot{x}\omega)\frac{dy}{ds} + \ddot{z}\frac{dz}{ds},$$

which reduces to $\ddot{s} - \omega^2 r\dfrac{dr}{ds}$, and if T be the sum of the tangential forces the equation of motion is

$$m\left(\ddot{s} - \omega^2 r\frac{dr}{ds}\right) = T.$$

If ρ be the radius of absolute curvature of the curve, the acceleration of the bead in direction of the principal normal is equal to

$$(\ddot{x} - \omega^2 x - 2\dot{y}\omega)\,\rho\frac{d^2x}{ds^2} + (\ddot{y} - \omega^2 y + 2\dot{x}\omega)\,\rho\frac{d^2y}{ds} + \ddot{z}\,\rho\frac{d^2z}{ds},$$

which reduces to

$$\frac{\dot{s}^2}{\rho} - \omega^2\rho\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2}\right) + 2\omega\rho\left(\dot{x}\frac{d^2y}{ds^2} - \dot{y}\frac{d^2x}{ds^2}\right),$$

and the expression, when multiplied by the mass of the bead, is equal to the sum of the forces in direction of the principal normal, and of the reaction in that direction.

In a similar manner the reaction of the curve in direction of its binormal can be determined.

154. *Motion of a heavy particle on a smooth inclined plane, the plane being in rigid connection with a fixed vertical axis and revolving uniformly.*

Take the line of greatest slope, drawn upwards, through the fixed point on the plane as the axis of x, and the normal to the plane as the axis of z. Then, referring to Art. (28),

$$\theta_1 = \omega\sin\alpha,\quad \theta_2 = 0,\quad \theta_3 = \omega\cos\alpha,$$

and therefore, z being zero,

$$u = \dot{x} - \omega y \cos \alpha,$$

$$v = \dot{y} + \omega x \cos \alpha,$$

$$w = \omega y \sin \alpha.$$

Taking the accelerations parallel to the axes, we have, if R be the reaction of the plane,

$$m(\dot{u} - v\omega \cos \alpha) = -mg \sin \alpha,$$

$$m(\dot{v} - w\omega \sin \alpha + u\omega \cos \alpha) = 0,$$

$$m(\dot{w} + v\omega \sin \alpha) = R - mg \cos \alpha.$$

The first two of these equations give

$$\ddot{x} - 2\omega \cos \alpha \dot{y} - \omega^2 \cos^2 \alpha x = -g \sin \alpha,$$

$$\ddot{y} + 2\,\omega \cos \alpha \dot{x} - \omega^2 y = 0,$$

thereby determining the motion, and the third equation gives the pressure. The integration is at once effected by aid of the calculus of operations, for the elimination of y leads to

$$\frac{d^4x}{dt^4} + \omega^2 \left(3 \cos^2 \alpha - 1\right) \frac{d^2x}{dt^2} + \omega^4 \cos^2 \alpha x = g\omega^2 \sin \alpha,$$

a linear equation with constant coefficients.

155. If the particle be constrained to move on a smooth curve in the revolving plane, the motion is determined by taking the resultant acceleration along the tangent to the curve.

This leads to the equation,

$$(\ddot{x} - 2\omega \cos \alpha \dot{y} - \omega^2 \cos^2 \alpha x) \frac{dx}{ds} + (\ddot{y} + 2\omega \cos \alpha \dot{x} - \omega^2 y) \frac{dy}{ds}$$

$$= -g \sin \alpha \frac{dx}{ds},$$

or $$\ddot{s} - \omega^2 \cos^2 \alpha x \frac{dx}{ds} - \omega^2 y \frac{dy}{ds} = -g \sin \alpha \frac{dx}{ds}.$$

Take for instance the case of a bead moving on a smooth circular wire which is made to revolve uniformly about a fixed vertical axis through its centre.

If θ be the angular distance of the bead from the axis of x, the preceding equation becomes

$$a\ddot{\theta} - \omega^2 a \sin^2 \alpha \sin \theta \cos \theta = g \sin \alpha \sin \theta,$$

and the angular motion is therefore given by

$$a\dot{\theta}^2 = \omega^2 a \sin^2 \alpha \sin^2 \theta - 2g \sin \alpha \cos \theta + C.$$

The radial pressure, R, of the bead, measured inwards, is given by the equation

$$ma(\dot{\theta}^2 + 2\omega\dot{\theta} \cos \alpha + \omega^2 \cos^2 \alpha \cos^2 \theta + \omega^2 \sin^2 \theta) = R + mg \sin \alpha \cos \theta.$$

156. The general problem of the motion of a particle on a smooth surface which is made to revolve about a fixed axis can be dealt with in a similar manner by aid of the general expressions for velocities and accelerations which are given in Art. (28).

Taking axes rigidly connected with the revolving surface, let the plane of zx contain the fixed axis about which the surface is revolving, and let α be the inclination of the axis of z to this fixed axis.

We have, then,

$$\theta_1 = \omega \sin \alpha, \quad \theta_2 = 0, \quad \theta_3 = \omega \cos \alpha,$$

and therefore,

$$u = \dot{x} - y\omega \cos \alpha, \quad v = \dot{y} - z\omega \sin \alpha + x\omega \cos \alpha,$$

$$w = \dot{z} + y\omega \sin \alpha.$$

The expressions for the accelerations are

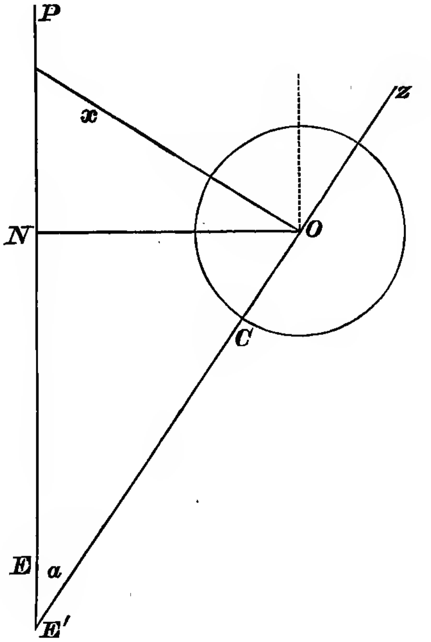
$$f_1 = \dot{u} - v\omega \cos \alpha,$$

$$f_2 = \dot{v} - w\omega \sin \alpha + u\omega \cos \alpha,$$

$$f_3 = \dot{w} + v\omega \sin \alpha,$$

and, if the acting forces are given, the equations of motion can be formed.

As a particular instance consider the motion of a pendulum, or, which is the same thing, of a particle inside a smooth sphere, rotating with the earth.

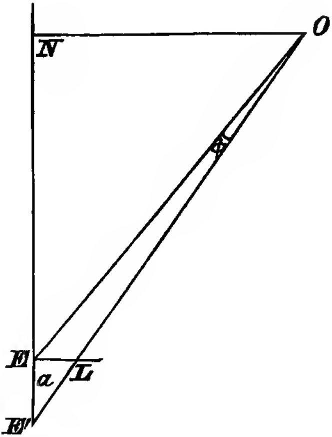


EP being the earth's axis, let C be the position of relative equilibrium of the particle.

Neglecting the size of the sphere in comparison with the distance OE, and regarding the earth as a sphere of which E is the centre, the direction $E'Oz$ is defined by the consideration that the resultant of the earth's attraction in the direction OE, and of the reaction at C, is the force $m\omega^2 ON$ in the direction ON.

As a matter of fact it has been shewn that, if c be the earth's radius, and g the acceleration due to gravity at the equator,

$$\omega^2 c = \frac{g}{289},$$



and therefore ω can be regarded as a very small quantity, and the displacement EOE' as an infinitesimal of the first order.

Hence if mg be the pressure at C, which is the weight of the particle,

$$\omega^2 . ON : g = EL : OL = \sin\phi : \cos(\alpha + \phi),$$

if the angle $OE'N = \alpha$,

or approximately, $\phi = \omega^2 c \sin\alpha \cos\alpha / g$,

where α is the colatitude of the point O.

Taking then $E'O$, that is the vertical at O, as the axis of z, and O as the origin, the expressions for the velocities and accelerations are those given above, with the exception of the expression for the velocity parallel to the axis of y, which now becomes,

$$v = \dot{y} - z\omega\sin\alpha + x\omega\cos\alpha - \omega c\sin\alpha.$$

The equations of motion are,

$$mf_1 = -R\frac{x}{a},\ \ mf_2 = -R\frac{y}{a},\ \ mf_3 = -R\frac{z}{a} - mg.$$

If the particle make very small excursions from its position of relative equilibrium, we can put $z = -a$, and neglect the squares of x and y. The approximate equations are then

$$\ddot{x} - 2\dot{y}\,\omega\cos\alpha = -\frac{R}{m}\frac{x}{a},$$

$$\ddot{y} + 2\dot{x}\,\omega\cos\alpha = -\frac{R}{m}\frac{y}{a},$$

$$2\dot{y}\omega\sin\alpha = -g + \frac{R}{m},$$

and eliminating R we obtain finally

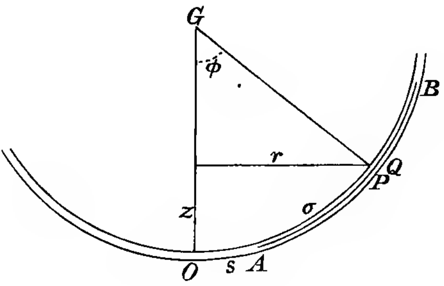
$$\ddot{x} - 2\dot{y}\,\omega\cos\alpha + \frac{g}{a}x = 0,$$

$$\ddot{y} + 2\dot{x}\,\omega\cos\alpha + \frac{g}{a}y = 0.$$

157. *Motion of a heavy chain inside a smooth tube which is revolving uniformly about a vertical axis in its own plane.*

Consider the motion of an element PQ, $(\delta\sigma)$ of the chain AB. Taking the arc $OA = s$, and $AP = \sigma$, the equation of motion is

$$m\delta\sigma\,\{\ddot{z}\sin\phi + (\ddot{r} - \omega^2 r)\cos\phi\} = \delta T - mg\delta\sigma\sin\phi.$$



The expression $\ddot{z}\sin\phi + \ddot{r}\cos\phi$ is what would be the tangential acceleration if there were no rotation and is therefore equal to $\ddot{s}$, and the equation becomes

$$m(\ddot{s} - \omega^2 r\cos\phi) = \frac{dT}{d\sigma} - mg\sin\phi;$$

or, analytically, σ not being a function of the time,

$$\dot{z} = s\sin\phi,\ \ \dot{r} = \dot{s}\cos\phi,$$

$$\ddot{z} = \ddot{s}\sin\phi + \dot{s}\dot{\phi}\cos\phi,\ \ \ddot{r} = \ddot{s}\cos\phi - \dot{s}\dot{\phi}\sin\phi,$$

whence
$$\ddot{z}\sin\phi + \ddot{r}\cos\phi = \ddot{s}.$$

Taking l as the length of chain AB, and integrating over the length l, we obtain

$$l\ddot{s} = \tfrac{1}{2}\omega^2(r'^2 - r''^2) - g(z' - z''),$$

r', z' being the co-ordinates of the end B, and r'', z'' of the end A.

All these four quantities being functions of s, we have an equation determining the motion of the chain in the tube.

If R be the rate of pressure at P in direction of the normal PG, per unit of length of chain, we have, by taking the acceleration in the direction PG,

$$m\delta\sigma\{\ddot{z}\cos\phi - (\ddot{r} - r\omega^2)\sin\phi\} = R\delta\sigma + \frac{T\delta\sigma}{\rho} - mg\delta\sigma\cos\phi,$$

or
$$\frac{m\dot{s}^2}{\rho} + \omega^2 r\sin\phi = R + \frac{T}{\rho} - mg\cos\phi.$$

For example, if the tube be circular in form, and if the arc OA subtend an angle θ at the centre, and AB an angle α,

$$a\alpha\ddot{\theta} = \tfrac{1}{2}\omega^2 a\{\sin^2(\alpha+\theta) - \sin^2\theta\} - g\{\cos\theta - \cos(\alpha+\theta)\},$$

and therefore,

$$a\alpha\dot{\theta}^2 = \tfrac{1}{4}\omega^2 a\{\sin 2\theta - \sin 2(\alpha+\theta)\} - 2g\{\sin\theta - \sin(\alpha+\theta)\} + C.$$

EXAMPLES.

1. Is a railway train heavier when going east or going west?

Shew that for a train weighing 180 tons, travelling 60 miles an hour in latitude 60°, the difference is about the weight of two men.

2. A particle of mass m is attached to one end of an elastic string, the other end of which is fastened to the vertex of a smooth cone of vertical angle 2α, having its axis vertical and vertex upwards; prove that the particle can move with a constant velocity v round the surface of the cone, and with the string stretched to double its natural length, provided that the modulus of elasticity $> mg \cos \alpha$, and that

$$2v^2 \cos \alpha < ag \sin^2 \alpha.$$

If the particle be slightly disturbed in direction of the string, find the time of a small oscillation.

3. A point describes a rhumb line on a sphere in such a way that its longitude increases uniformly; prove that the resultant acceleration varies as the cosine of the latitude, and that its direction makes with the normal an angle equal to the latitude.

4. A material particle rests on a rough plane inclined at a given angle to the horizon, the plane begins to rotate round an axis perpendicular to it, with a velocity commencing from zero and continually increasing. Determine the velocity at which the particle will commence to move on the plane, and the condition that the commencement of the motion is simultaneous with that of the plane.

5. A particle moves under the action of gravity on a smooth surface of revolution whose axis is vertical; shew that its path cannot be a geodesic, unless it be a meridian section.

6. A smooth right circular cylinder is placed with its axis horizontal, and a particle moving uniformly with velocity v along the lowest generating line receives a horizontal impulse at right angles to this line and just sufficient to carry it to the highest point of the cylinder. If the particle be prevented from leaving the cylinder, shew that its subsequent path is such that if the cylinder be developed its equation is

$$y = \pi a - 4 \tan^{-1} \epsilon^{-\frac{x}{v}\sqrt{\frac{g}{a}}},$$

and that the highest generating line is an asymptote to the curve.

7. Determine the motion of a heavy particle moving in a smooth tube in the form of a circle, which revolves uniformly about a vertical diameter.

8. In the centre of a hollow sphere resides a repulsive force. A heavy particle is projected horizontally along the surface of the sphere from a point distant 60° from the highest point with a velocity due to falling through the diameter by its weight only. Shew that it will be again moving horizontally at a point whose distance from the lowest point is $\tan^{-1}\sqrt[4]{\frac{3}{4}}$.

9. A particle slides on a smooth helix of radius a and angle α under the action of a force to a fixed point in the axis equal to μ times the distance. Investigate the motion, and prove that the pressure cannot vanish unless the greatest velocity of the particle be $\sqrt{\mu a} \sec \alpha$.

10. A heavy particle moves in a smooth sphere; shew that, if the velocity be due to the level of the centre, the pressure on the surface will vary as the depth below the centre.

11. Two equal particles attracting one another with a force, the accelerating effect of which is $\rho^2 \times$ distance, are placed in two rough straight tubes at right angles to one another, and the friction is equal to the pressure in each tube; prove that, if they be initially at unequal distances,

one moves for a time $\frac{\pi}{2\rho}$ before the other begins to move, and that, while they are approaching the point of intersection of the tubes, they move in the same manner as the projections of the two extremities of a diameter of a circle upon a straight line on which the circle rolls.

12. A particle is revolving on a smooth plane about a centre of force, the accelerating effect to the centre being $\mu \times$ distance, and when the body arrives at an apse the plane begins to revolve with an angular velocity $\frac{1}{2}\sqrt{3\mu}$ about the apsidal line; shew that the subsequent orbit described on the plane will be a portion of a parabola; and that, when the particle leaves the plane, its velocity will be $\sqrt{3} \times$ velocity at the vertex.

13. A smooth parabolic tube whose latus rectum is $4a$ rotates about its axis which is vertical, the vertex being downwards, with uniform angular velocity ω. Find ω in order that a heavy particle may be in equilibrium at any point of the tube.

If the angular velocity of the tube be greater than this and a particle be projected down the tube from any point with velocity just sufficient to make it reach the vertex, shew that the equation to the projection of the subsequent path, on a horizontal plane, is

$$\theta = C - \log \frac{r}{2a + \sqrt{4a^2 + r^2}} - \frac{\sqrt{4a^2 + r^2}}{2a}.$$

14. A heavy particle, in contact with the lower half of the internal surface of a fixed smooth spherical shell, is projected horizontally with velocity V, the radius through the particle making initially an angle α with the vertical. Find the pressure in terms of the velocity at any time and prove that if $V^2 \cos \alpha < 2ag$, the particle cannot leave the surface, but that if $V^2 \cos \alpha > 2ag$, it may do so, and find the additional condition necessary.

15. A heavy bead moves along a vertical circular wire which revolves about a vertical straight line in its own plane. Find the time of a small oscillation, and the resistance on the wire.

16. A small bead slides on a smooth circular ring of radius a, which is made to revolve round a vertical axis passing through its centre with uniform angular velocity ω, the plane of the ring being inclined at a constant angle α to a horizontal plane. Shew that the law of angular motion of the bead on the ring is the same as that of a bead on a ring of radius $a \operatorname{cosec} \alpha$ revolving round a vertical diameter with angular velocity $\omega \sin \alpha$.

17. A smooth wire, in the form of a parabola, latus rectum l, revolves about its axis which is vertical, the vertex being uppermost, with uniform angular velocity $= \sqrt{g/l}$; a string, passing through a fixed ring at the focus carries, at one end, a small ring, mass m, which slides on the wire, and at the other end a particle, mass m', which hangs freely. Given the velocity, V, of the ring at the vertex, determine the rate at which the ring describes the parabola at any point.

If $m = 4m'$, and $2V^2 = gl$, prove that, at a time t after the ring has passed the vertex, the angle θ between the two parts of the string is given by the equation

$$\cot \frac{\theta}{2} = \sqrt{\frac{2g}{l}}\, t.$$

18. Three masses m_1, m_2, m_3 are fastened to a string which passes through a ring, and m_1 describes a horizontal circle as a conical pendulum while m_2 and m_3 hang vertically.

If m_3 drop off, prove that the instantaneous change of tension of the string is

$$\frac{gm_1m_3}{m_1 + m_2}.$$

19. A particle is placed between two smooth co-axial circular cylinders of nearly equal radii, whose common axis is inclined to the vertical, and slides down under gravity. If β be the angular distance of its initial position from the lowest point of the cross-section through that position, shew that the particle will never press the inner surface if $2\beta < \pi$; but if $2\beta > \pi$, the particle will pass from the inner to the outer surface, and back again, and so on, when ϕ, its angular distance at any point from the lowest point of the cross-section through that point, takes the successive values given by $3 \cos \phi = 2 \cos \beta$.

20. There are two points P and Q which move so that the line of motion of each relative to the other is always parallel to a given direction. If the motion of P, and the initial position of Q be given, shew how to determine the surface on which it must move. If the orbit of P be plane, prove that this surface is a cylinder. If the motion of P be that of a projectile in vacuo, and the relative velocity of P and Q constant, determine the motion of Q.

21. A smooth hollow ellipsoid of revolution is fixed with its axis ($2a$) vertical, and a particle is projected from a point in the horizontal plane through the centre and on the inside surface with a velocity $\sqrt{2ga}$ and inclination α to the horizon. Find α in order that the greatest depth below the centre may be $2a/3$, and find in that case the greatest height reached.

22. A smooth cylinder, whose transverse section is a cycloid generated by a circle of diameter a, is placed with its axis horizontal, the axis of the cycloidal section being vertical and its vertex downwards. A heavy particle is allowed to fall from rest at any point of the surface and is attracted by a perfectly elastic plane perpendicular to the axis of the cylinder, with a force varying directly as the distance from the plane, whose absolute intensity is $2g/a$. Shew that the path of the particle will be such that if the cylinder be developed it will develope into successive portions of a parabola.

23. A smooth surface is generated by the revolution of the curve $x^2y = c^3$ about the axis of y which is vertically downwards, and a heavy particle is projected along the surface with velocity due to the depth below the horizontal plane through the origin: prove that its path intersects all the meridians at a constant angle.

24. A surface of revolution is such that if it be held with its axis vertical, and a heavy particle be projected along it with suitable velocity at any point in any direction, its path will cut every meridian of the surface at a constant angle. Shew that the surface may be generated by the revolution round the axis of y of the curve

$$h(x^2 - a^2) + x^2y = 0.$$

25. A parabolic wire, axis vertical and vertex downwards, rotates about its axis with uniform angular velocity. A ring slides down it under gravity; prove that it may descend with constant velocity.

26. A heavy string of given length is enclosed in a smooth straight tube, which is made to revolve uniformly about a vertical axis, so as to describe a right circular cone; determine the motion of the string and the tension at any point.

27. A surface is of the form traced out by the revolution of the curve $z = c \cos x/c$ about the axis of z: the surface being placed with its axis vertical, a particle is projected upon it in such a manner that it describes a horizontal circle in a given time t. Prove that the number of possible circles is even, except in that case in which the time of revolution satisfies the equation

$$1 + \frac{gt^2}{4\pi^2 c} \cdot \cos \sqrt{\left(\frac{g^2 t^4}{16\pi^4 c^2} - 1\right)} = 0.$$

28. A heavy particle moves on a curve which revolves uniformly about a vertical axis; prove that the time of an oscillation of the particle about a position of relative equilibrium will be

$$\frac{2\pi}{\omega}\left(\frac{\rho \sin \alpha}{k - \rho \sin \alpha \cos^2 \alpha}\right)^{\frac{1}{2}},$$

ρ being the radius of curvature at the point of equilibrium, α the angle made by the normal at that point with the vertical, k the distance of the point from the axis of revolution, and ω the angular velocity of the curve.

29. An anchor ring is formed by the revolution of a circle of radius (c) about an axis in its own plane, distant (a) from the centre of the circle. A particle is projected along the equator, of smaller radius with velocity (v), and is acted on by a centre of attractive force in the centre of the axis, and equal at distance r to μr^n; shew that if the particle be slightly displaced it will continue to return to its original path at equal angular intervals (θ), where

$$\left(\frac{\pi}{\theta}\right)^2 = \frac{a-c}{c}\left\{\frac{\mu a (a-c)^n}{v^2} - 1\right\}.$$

30. A particle constrained to move in the surface of a smooth ellipsoid is under the attraction of an internal ellipsoidal shell, the two surfaces being confocal; prove that if the particle be projected from an umbilicus with a given velocity, it will return to the umbilicus in a time which is independent of the direction of projection.

31. Two infinite straight lines which are at right angles but do not meet attract according to the law of gravitation. Prove that, if a particle be projected from the middle point of the shortest distance between the lines in direction of the line bisecting the angle between them, it will continue to move in a straight line: and find the limits of the motion. Prove also that a particle will move with uniform velocity, under the attraction of the lines, in any smooth tube which takes the form of the curve of intersection of a certain hyperbolic paraboloid with any one of a certain series of oblate spheroids.

32. A small smooth groove is cut on the surface of a right cone, axis vertical and vertex upwards, in such a manner that the tangent is always inclined to the vertical at the same angle β. A particle slides down the groove from rest at the vertex; shew that the time of descending a vertical height h is equal to the time of falling freely through a height $h\sec^2\beta$. Shew also that the pressure is constant and that it makes a constant angle θ with the principal normal to the path, such that $2\tan\theta\sqrt{\cos^2\alpha-\cos^2\beta}=\sin\alpha$, 2α being the angle of the cone.

33. A shot, fired at an angle of elevation (α), strikes a point at a horizontal distance (d) from the point of projection; shew that if any subsequent shot falls within a small distance (c) of the point first struck, then the direction of projection must meet the unit sphere drawn with its centre at the point of projection within a small ellipse whose equation when referred to certain axes is very approximately

$$d^2(x^2\cos^2 2\alpha+y^2\cos^2\alpha)=c^2\sin^2\alpha\cos^2\alpha.$$

34. Three particles of equal mass which attract one another according to the law of the inverse square, are free

to slide on three wires which form the edges of a prism whose base is an equilateral triangle. If the system is slightly disturbed from its position of equilibrium, prove that it executes a small oscillation in the time $2\pi\sqrt{a^3/3m}$; m being the mass of a particle and a the mutual distance of the wires.

35. A particle is free to move along a helix whose axis is vertical, and a centre of force whose accelerating effect is $\mu\times$ distance resides in the axis of the helix. The particle is so placed as to be in equilibrium, and the centre of attraction then begins to move vertically upwards with a velocity V; prove that after a time t

$$\mu\sin^2\alpha\,(s\sin\alpha - Vt)^2 + (\dot{s}\sin\alpha - V)^2 = V^2,$$

s being the arc of the helix measured from the position of equilibrium, and α the angle which the helix makes with the horizon. Hence determine s in terms of t.

36. A circular tube of smooth bore has its centre fixed above a rough horizontal plane and is made to roll uniformly in contact with the plane. Shew that the motion of a particle of unit mass within the tube is given by

$$a\ddot{\phi} - a\Omega^2\sin^2\alpha\sin\phi\cos\phi + g\sin\alpha\cos\phi = 0,$$

and the pressures towards the centre and perpendicular to the plane of the tube are determined by

$$a(\dot{\phi} + \Omega\cos\alpha)^2 + a\Omega^2\sin^2\alpha\sin^2\phi + g\sin\alpha\cos\phi = R,$$

$$2a\dot{\phi}\Omega\sin\alpha\cos\phi + a\Omega^2\sin\alpha\cos\alpha\cos\phi - g\cos\alpha = S,$$

where Ω is the angular velocity of the point of contact round the vertical and α the inclination of the plane of the tube to the horizon.

CHAPTER X.

THE HODOGRAPH AND THE BRACHISTOCHRONE.

158. *The Hodograph. If from any fixed point a straight line be drawn parallel to the direction of motion of a moving point and of a length proportional to the velocity of the point, the locus of its extremity is the hodograph of the path of the point.*

Polar equation of the hodograph.

If θ be the inclination, to any fixed direction, of the tangent to the path, and if the velocity $= f(\theta)$,

then $$r = cf(\theta)$$

is the polar equation of the hodograph, c being any constant.

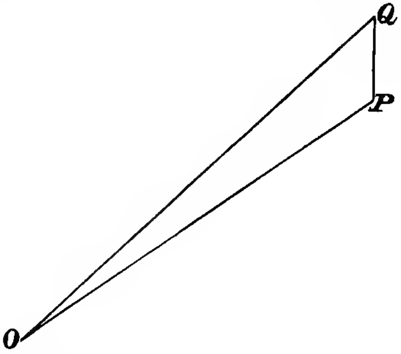
For example, if a heavy particle slide down the arc of a smooth vertical circle from its highest point, the hodograph is

$$r^2 = 2gc\,(1 - \cos\theta).$$

Again, if a particle describe an ellipse under the action of a force to its centre, $v \propto CD$, Art. (105), and therefore the ellipse is its own hodograph.

159. If OP and OQ represent, in direction and magnitude, the velocities of a particle of the times t and $t + \delta t$, PQ represents, by the triangle of velocities, the velocity imparted during the time δt, and therefore if f be the acceleration of

the particle, PQ is the direction of the acceleration, and its length $= f\,\delta t$.



Hence it follows that the tangent to the hodograph is the direction of the acceleration, and that, if σ be the arc of the hodograph, $f = \dot{\sigma}$, that is, the velocity in the hodograph is equal to the acceleration of the particle.

If for instance a particle move in a plane curve under the action of a force making a constant angle with the direction of motion, the hodograph is an equiangular spiral.

In general, if x, y, z be the co-ordinates of a particle in motion, and ξ, η, ζ the co-ordinates of the hodograph, we have

$$\xi = \dot{x}, \quad \eta = \dot{y}, \quad \zeta = \dot{z},$$

and from these the equations of the hodograph can be found.

Thus, if a heavy particle slide down a smooth helix, the axis of which is vertical,

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\theta\tan\alpha,$$

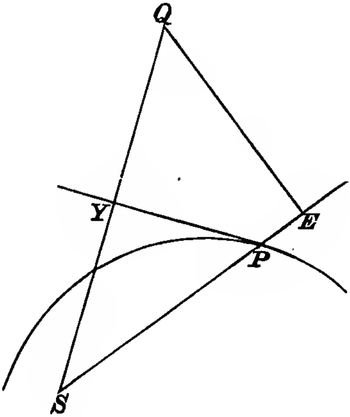
and $$2gz = v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = a^2\sec^2\alpha\,\dot{\theta}^2;$$

$$\therefore \quad \xi = \sin\theta\cos\alpha\sqrt{2ga\theta\tan\alpha}, \quad \eta = \cos\theta\cos\alpha\sqrt{2ga\theta\tan\alpha},$$

and $$\zeta = \sin\alpha\sqrt{2ga\theta\tan\alpha}.$$

Hence $\xi^2 + \eta^2 = \zeta^2\cot^2\alpha$, shewing that the hodograph is a curve drawn on the surface of a right cone, a result which presents itself at once from the geometry of the case.

160. In the particular case of central forces, the hodograph is the reciprocal polar of the path, turned through a right angle.



For, if $SQ = V = h/p$, SQ will be the radius vector of the hodograph, turned through a right angle. Or, which is the same thing, the hodograph is the inverse of the pedal curve turned through a right angle.

Hence for a conic section described under the action of a force to the focus, the hodograph is a circle.

If $p = f(r)$ be the equation of a central orbit, the equation of the hodograph is

$$\frac{c^2}{r'} = f\left(\frac{c^2}{p'}\right).$$

For QE, the perpendicular on SP, is the tangent to the path of Q, and

$$p = SY = \frac{c^2}{SQ} = \frac{c^2}{r'}, \quad r = SP = \frac{c^2}{SE} = \frac{c^2}{p'}.$$

In all cases of free motion under the action of parallel forces the hodograph is obviously a straight line.

161. Conversely, if the hodograph and its mode of description be known, the path can be determined.

Suppose for instance the hodograph to be a helix described with uniform velocity. We then have,

$$\dot{x} = a \cos \omega t, \quad \dot{y} = a \sin \omega t, \quad \dot{z} = a \tan \alpha \,.\, \omega t,$$

and the integration of these equations gives the equations of the path.

From the first two we obtain the form

$$(x - \alpha)^2 + (y - \beta)^2 = c^2,$$

so that the path is a curve on the surface of a cylinder.

162. If two particles describe the same curve, in the same direction, under the action of the same central force, the chord of the hodograph corresponding to their positions at any time represents the velocity of either relative to the other.

For instance, if two particles describe the curve, $r \sin 3\theta = a$, under the action of a force to the origin, the hodograph is a three-cusped hypocycloid.

Now it is a known property of this hypocycloid that any tangent to it, bounded by the curve, is of constant length.

Hence we infer that whenever the directions of motion of the two particles meet on the curve the velocity of either, relative to the other, is always the same.

The Brachistochrone.

163. The Brachistochrone is the curve along which a particle can be guided in a given field of force from one given point, or from one given curve or surface to another, so as to make the transit in the least possible time. We shall consider first some special cases and afterwards prove some general characteristics of brachistochrones.

To find the brachistochrone for the case of a heavy particle in a vertical plane from one given point to another.

Measuring y downwards from the starting-point

$$v^2 = 2gy,$$

and the expression

$$\int \frac{ds}{v}, \quad \text{or} \int \frac{\sqrt{1+p^2}\,dx}{\sqrt{2gy}} \text{ is to be a minimum.}$$

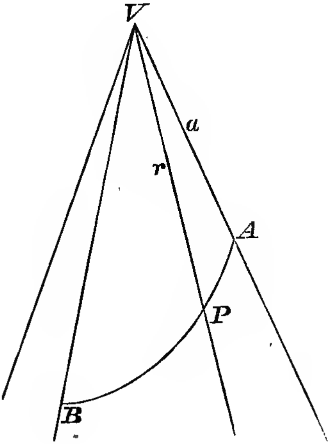
Employing the ordinary processes of the Calculus of Variations, we obtain

$$\frac{\sqrt{1+p^2}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{1+p^2}} + C, \quad \text{or} \quad \frac{dx}{dy} = \sqrt{\frac{y}{2c-y}}.$$

Hence $$x = c \operatorname{vers}^{-1}\frac{y}{c} - \sqrt{2cy - y^2}$$

is the brachistochrone, and this represents a cycloid having its cusp at the origin.

164. *A heavy particle moves on the surface of a smooth circular cone, axis vertical and vertex upwards; it is required to find the brachistochrone from a given point to a given generating line.*



If A be the starting-point, $VA = a$ and $VP = r$,

$$v^2 = 2g \cos \alpha \, (r - a),$$

and the expression,

$$\int_0^\beta \sqrt{\frac{\left(\frac{dr}{d\phi}\right)^2 + r^2 \sin^2 \alpha}{r - a}} \, d\phi,$$

where ϕ is the azimuthal angle is to be a minimum, from which condition we obtain

$$\left(\frac{dr}{d\phi}\right)^2 = r^2 \sin^2 \alpha \, \frac{r^2 \sin^2 \alpha - c\,(r - a)}{c\,(r - a)},$$

as the differential equation of the brachistochrone.

At the limit β, employing the boundary equation, we find that $\dfrac{dr}{d\phi} = 0$, and therefore that the brachistochrone is horizontal at its lower extremity.

The fact that the curve passes through the point $(a, 0)$ theoretically determines c, and the radius to the lowest point is given by the positive root of the equation,

$$r^2 \sin^2 \alpha - c\,(r - a) = 0.$$

165. *A particle moves under the action of a repulsive force from a fixed point O varying as the distance; and starts with the velocity $\sqrt{\mu} \, . \, OA$ from the point A.*

To find the brachistochrone to another point B, we first observe that, at a distance r,

$$v^2 = \mu r^2,$$

and therefore

$$\int \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{r} \, d\theta$$

is a minimum, the limits being constant.

This leads to

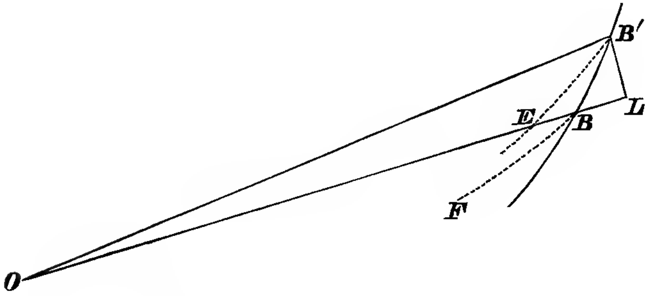
$$r \frac{d\theta}{ds} = C,$$

shewing that the brachistochrone is an equiangular spiral.

If it be required to determine, in this case, the brachistochrone from the fixed point A to a given curve, $r=f(\theta)$, we have, in addition to the equation, $rd\theta = Cds$, the boundary condition.

Taking r_1, θ_1 as the co-ordinates of the bounding point B on the curve $r=f(\theta)$, this condition is

$$\frac{1}{r_1}\sqrt{r_1^2+\left(\frac{dr}{d\theta}\right)_1^2}\,d\theta_1+\frac{\frac{1}{r_1}\left(\frac{dr}{d\theta}\right)_1}{\sqrt{r_1^2+\left(\frac{dr}{d\theta}\right)_1^2}}\,\delta r_1=0.^*$$



Now, if BB' be a small arc of the given curve and $BOB' = d\theta_1$, and if the dotted line BF represent part of the brachistochrone, and $B'E$ the slightly variated curve, meeting in E the radius vector OB, then $\delta r_1 = -BE$.

From the figure it will be seen that

$$BE = EL - BL = B'L \cot EB'O - B'L \cot BB'O,$$

and therefore $$\delta r_1 = \left\{f'(\theta_1) - \left(\frac{dr}{d\theta}\right)_1\right\} d\theta_1.$$

Substituting in the above equation, and observing that $d\theta_1$ is an arbitrary quantity, we obtain

$$r_1\left(\frac{d\theta}{dr}\right)_1 \cdot \frac{r_1}{f'(\theta_1)} + 1 = 0, \;\ldots\ldots(A),$$

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* See Todhunter's *Integral Calculus*, Art. (358).

proving that the brachistochrone must intersect the given curve at right angles.

To complete the solution, we obtain from $rd\theta = Cds$, the equation

$$r = a\epsilon^{\theta \cot \alpha}, \text{ where } a = OA,$$

and, to find r_1, θ_1 and α, we have the above equation (A), with the equations,

$$r_1 = f(\theta_1),\ r_1 = a\epsilon^{\theta_1 \cot \alpha}.$$

166. *The brachistochrone for a particle moving in a given field of force.*

The system of the particle and the field, being a conservative system, as in Art. (143), we know that the velocity depends upon the position of the particle, and therefore, if v be its velocity when passing through the point (x, y, z),

$$v = f(x, y, z).$$

For the brachistochrone we have to make the expression

$$\int \frac{ds}{v} \text{ or } \int \frac{\sqrt{1 + p^2 + q^2}}{v}\, dx$$

a minimum, p and q standing for dy/dx and dz/dx.

The methods of the Calculus of Variations (see Todhunter's *Integral Calculus*, Art. 364) lead to the equations,

$$\left.\begin{aligned} \sqrt{1 + p^2 + q^2}\, \frac{d}{dy}\left(\frac{1}{v}\right) - \frac{d}{dx}\left(\frac{1}{v} \frac{p}{\sqrt{1 + p^2 + q^2}}\right) = 0, \\ \sqrt{1 + p^2 + q^2}\, \frac{d}{dz}\left(\frac{1}{v}\right) - \frac{d}{dx}\left(\frac{1}{v} \frac{q}{\sqrt{1 + p^2 + q^2}}\right) = 0, \end{aligned}\right\} (A);$$

which reduce to

$$v \frac{d^2 y}{ds^2} - \frac{dv}{ds}\frac{dy}{ds} + \frac{dv}{dy} = 0,$$

$$v \frac{d^2 z}{ds^2} - \frac{dv}{ds}\frac{dz}{ds} + \frac{dv}{dz} = 0.$$

Multiplying by $\frac{dy}{ds}$ and $\frac{dz}{ds}$ respectively, and adding the results, we obtain the symmetrical equation

$$v\frac{d^2x}{ds^2} - \frac{dv}{ds}\frac{dx}{ds} + \frac{dv}{dx} = 0.$$

Any two of these three equations determine the brachistochrone.

167. *The brachistochrone for a particle constrained to move on a given smooth surface.*

Taking K and L to represent the left-hand members of the equations (A), we have in this case

$$\int_{x_0}^{x_1} (K\delta y + L\delta z)\, dx = 0;$$

and also, if $\phi(x, y, z) = 0$ be the given surface,

$$\frac{d\phi}{dy}\delta y + \frac{d\phi}{dz}\delta z = 0.$$

We hence obtain the single condition

$$\frac{K}{\frac{d\phi}{dy}} = \frac{L}{\frac{d\phi}{dz}};$$

and this equation, with $\phi(x, y, z) = 0$, determines the brachistochrone.

168. In the case of a particle moving in a field of force, if v be the velocity of the particle, and V the potential energy, the equation of energy is

$$\tfrac{1}{2}mv^2 + V = C.$$

From this equation we obtain

$$mv\frac{dv}{dx} = -\frac{dV}{dx} = mX,$$

if mX be the acting force.

The equations of Art. (166) now become

$$v^2\frac{d^2x}{ds^2} - v\frac{dv}{ds}\frac{dx}{ds} + X = 0,$$

$$v^2\frac{d^2y}{ds^2} - v\frac{dv}{ds}\frac{dy}{ds} + Y = 0,$$

$$v^2\frac{d^2z}{ds^2} - v\frac{dv}{ds}\frac{dz}{ds} + Z = 0.$$

If λ, μ, ν be the direction cosines of the binormal we obtain

$$\lambda X + \mu Y + \nu Z = 0;$$

and hence it follows that

the osculating plane contains the resultant of the acting forces.

Again, multiplying by the direction cosines (l, m, n) of the principal normal, and adding, we find that

$$\frac{v^2}{\rho} + lX + mY + nZ = 0;$$

that is, the component of the acting force in direction of the principal normal is equal to $-mv^2/\rho$.

Now, for free motion, the force along the principal normal is equal to mv^2/ρ.

If then the normal force be reversed in direction, the tangential force remaining unchanged, a free path becomes a brachistochrone, and the converse is equally true.

In other words the forces are reflections, or images, of each other with regard to the tangent, both in direction and magnitude.

This theorem is due to Professor Townsend, and is given with illustrations in Vol. XIV. of the *Quarterly Journal of Mathematics.*

For instance, if a particle move in the curve, $s = 4a \sin\phi$, under the action of a force inclined to the direction of motion at the angle $\pi/2 + \phi$, it will be found that the force is constant.

The image of this case is the cycloidal brachistochrone under the action of gravity.

For another example take the case of an ellipse described freely under the action of forces to the two foci, each varying inversely as the square of the distance.

The same ellipse will be a brachistochrone for repulsive forces from the two foci, each varying inversely as the square of the distance from the other focus.

EXAMPLES.

1. A point moves in a straight line under the action of a force varying as the distance from a point in that line; prove that the corresponding point in the hodograph moves as though acted upon by a similar force.

2. One particle describes a given orbit about a centre of force, and another particle describes the hodograph of that orbit under the action of a force in the pole of the hodograph, shew that the product of the accelerations of the particles at two corresponding points of their orbits varies as the product of the central distances of those points.

3. If P and Q be the tangential and normal forces, and ϕ the inclination of the tangent to a fixed direction, the hodograph is

$$\log \frac{r}{a} = \int_0^\theta \frac{P}{Q}\, d\phi.$$

4. A smooth elliptic tube is placed with its major axis vertical and a particle allowed to slide down it, starting from rest at the highest point; shew that the hodograph is given by the equation

$$r = c \sin \frac{1}{2} \left\{ \cot^{-1} \left(\frac{a}{b} \cot \theta \right) \right\}.$$

5. Prove that the hodograph of a central orbit can itself only be a central orbit under the action of a force to the origin from which its radii are drawn when the central orbit is an ellipse or hyperbola.

6. A heavy particle moves on a rough curve in a vertical plane so that the pressure on the curve is constant. Prove that its hodograph is a conic described as about a centre of force in the focus.

7. If a particle describe a lemniscate under the action of a force to the pole, prove that the hodograph is of the form

$$r^2 = a^2 \sec^3 \frac{\pi - 2\theta}{3}.$$

8. If a particle move in a brachistochrone in an open field of force the pressure on the constraining curve is $2mv^2/\rho$.

9. A rough tube in the form of a cycloid is placed with its axis vertical and vertex upwards. A heavy particle is projected along the tube from the vertex with a given velocity V, find the velocity in any subsequent position.

If the coefficient of friction be $\tan \lambda$, and the initial velocity be to that which would be acquired in descending freely down the tube, supposed smooth, as $\sin \lambda : 1$, prove that the hodograph is a circle.

10. Find the hodograph in the cases of motion in a cardioid under the action of a force to the cusp.

11. One circle rolls uniformly on the circumference of another, on the outside; find the hodograph of a point on the circumference of the rolling circle.

12. Find the hodograph in the cases of the motion of a heavy particle on a smooth cycloid, the axis of which is vertical and the vertex (1) upwards, (2) downwards.

13. A particle is moving under the action of a force perpendicular to and proportional to the distance from the line of zero velocity, shew that the brachistochrone is a circle.

14. Prove that a parabola is a brachistochrone,

(1) for a constant force from the focus;

(2) for a force from the directrix varying inversely as the square of the distance from the directrix.

15. Prove that if the force vary inversely as the cube of the distance from a fixed point, the brachistochrone will be an equilateral hyperbola.

16. Shew that the parabola is brachistochronous for a force acting perpendicularly from its axis, and varying directly as the axial and inversely as the square of the focal distance, the line of no velocity coinciding with the axis.

17. A particle moves in a vertical plane in a medium whose resistance is kv^n: determine the hodograph. Shew that it will be an algebraic curve if n be an odd integer.

Defining the instantaneous parabola as the parabola that would be described if at any instant the resistance cease to act; shew that the vertex of such a parabola is at any instant moving downwards at an angle $\tan^{-1}(\frac{1}{2}\tan\phi)$ to the horizon, where ϕ is the angle the particle's path makes with the horizon.

18. Two particles are describing free paths in one plane which are hodographs to one another; if the particles be always at corresponding points, prove that the paths must be conic sections, and find the nature of the forces acting on the particles.

19. A body moves on a right circular cone, the velocity varying as the nth power of the cosine of the angle of inclination to the vertical; and the body moves along the curve of quickest descent from one given point to another. Shew that, if the cone be developed, the path will become a curve such that the perpendicular on the tangent varies as some power of the polar subtangent; and find the curves for the cases $n = 1$ and $n = 0$.

20. If the velocity of a carriage along a road is proportional to the cube of the cosine of the inclination of the road to the horizon, determine the path of quickest ascent from the bottom to the top of a hemispherical hill, and shew that it consists of a spherical curve described by a point of a great circle which rolls on a small circle described about the pole with a radius $\frac{\pi}{6}$, together with an arc of a great circle. How is the discontinuity introduced into this problem?

21. Prove that any curve which is a free path for a force to a fixed centre is also a brachistochrone for an equal force enveloping its caustic by reflexion from the fixed centre as focus.

22. If the particle move in the brachistochrone between two given points on a surface of revolution, prove that the area swept over by the radius vector on the horizontal plane is proportional to the action.

23. A projectile moves under gravity in a uniform medium whose resistance varies as the velocity. Prove that the hodograph of the trajectory is a straight line, and that the velocity of the point on the hodograph is proportional to the horizontal velocity of the projectile.

24. Find the hodograph of an elliptic orbit described under the action of a force to the focus, and hence prove that the mean value, taken with regard to time, of the inverse square of the radius vector is equal to the product of the reciprocals of the semiaxes.

25. A particle moves freely under the action of a force whose direction is always parallel to a fixed plane, and describes a curve which lies on a right circular cone and crosses the generating lines at a constant angle, prove that its hodograph is a conic section.

26. A heavy particle is projected from a given point along a smooth groove cut on the surface of a right circular cone, whose axis is vertical and vertex upwards, with the velocity due to the depth from the vertex. Prove that, if it reach another given point not more than half way round the cone in the least possible time, the curve of the groove must be such as would if the cone were developed become a parabola with the point corresponding to the vertex as focus.

27. A particle, acted on by a central attractive force whose accelerating effect at a distance r is $\frac{\mu r}{(a^2+r^2)^2}$, a being a constant, is projected from a given point with the velocity from infinity; prove that the form of the groove, in which it

must move in order to arrive at another given point in the shortest possible time, is an hyperbola whose centre coincides with the centre of force.

28. A particle is constrained to move on a surface of revolution under the action of forces the directions of which pass through the axis, and which depend upon the distances from the axis or from fixed points in it; find the differential equation, (in r and θ, or in any other form,) to the projection, on a plane perpendicular to the axis, of the brachistochronous path between two points on the surface, and prove that the velocity at any point is proportional to the distance from the axis, and to the sine of the angle between the path and the generating curve through the point.

If the surface be a hemisphere, and the force be attractive and vary as the distance from the axis, it may be shewn that when the starting-point is in the rim of the surface, the projection is a straight line.

29. Find the differential equations of the brachistochrone on the surface of a sphere which is rotating round a diametral axis with uniform angular velocity.

Prove that if r and θ be the co-ordinates of the projection of the particle on a plane perpendicular to the axis, $\dot{r}$ will be proportional to the resolved part of the force of constraint perpendicular to the meridian plane of the particle at any instant, and that under certain conditions the equation between r and θ assumes the form

$$a = r \cosh m\theta.$$

30. The velocity of the current in a stream is proportional to the distance from the bank. A man who swims at a given rate wishes to get as far as possible down the river in a given time; shew that he must start from the bank at an angle whose tangent is proportional to that time.

CHAPTER XI.

MOTION OF TWO PARTICLES ACTING ON EACH OTHER.

169. If two particles, attracting each other, move in a plane, we know that their centre of gravity is either at rest or is in motion with a velocity constant in magnitude and direction.

Having given the velocities at any instant we can find the velocity of the centre of gravity, and by reversing this velocity on the whole system we find the velocities of the two particles relative to their centre of gravity, and the case is then reduced to that of a central force.

In all cases the force of each particle on the other is proportional to the product of the masses and a function of the distance, and therefore by a proper choice of units is represented by the expression $mm'\phi(r)$.

If m, m' be the masses of the particles, and u, u' their initial velocities relative to their centre of gravity, these velocities are in parallel and opposite directions, and are such that

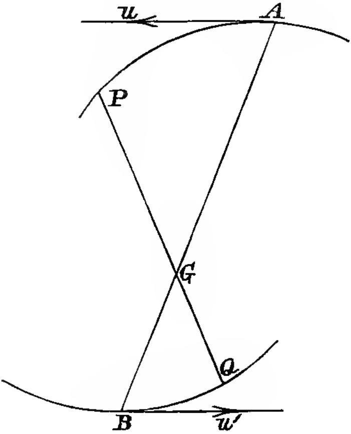
$$mu = m'u'.$$

It is evident that the two paths about G are similar curves.

If $PQ = r$, and if $mm' \phi(r)$ be the force exerted by each particle on the other, then, considering the motion of P, this force

$$= mm' \phi \left\{ \frac{m + m'}{m'} GP \right\},$$

which is a function of the distance GP, and the motion is therefore determined as in Chapter VII.



If the path of P with regard to G be determined in the form, $GP = f(\theta)$, the path of P relative to Q is given by the equation,

$$\frac{m'}{m + m'} PQ = f(\theta).$$

Or, we can determine the path of P relative to Q by finding the initial velocity of P relative to Q, and observing that the acceleration of P relative to Q is

$$(m + m') \phi(PQ),$$

which again reduces the case to that of a force to a fixed centre.

170. *Motion of two particles in a plane attracting each other with a force varying as the distance.*

The force on $P = mm' \, PQ = m\,(m+m')\,GP$, and the acceleration of P in the direction $PG = (m + m')\,GP$, hence it follows that P describes an ellipse about G as centre in the time $2\pi/\sqrt{m+m'}$, and that, if u be the initial velocity of P relative to G, the semidiameter conjugate to $GA = u/\sqrt{m+m'}$.

If u, u' be the initial velocities of P and Q relative to G, so that $mu = m'u'$, the initial velocity of P relative to Q

$$= u + u' = (m + m')\,u/m',$$

and the acceleration of P relative to $Q = (m + m')\,PQ$, so that the periodic time is $2\pi/\sqrt{m+m'}$, and the semidiameter, conjugate to AB, of the relative path

$$= u\sqrt{m+m'}/m'.$$

These last results are of course at once derivable by geometry from the preceding.

171. *Motion of two particles in a plane when the law of attraction is that of the inverse square of the distance.*

Taking the force between the particles to be $mm'/(\text{Distance})^2$, the acceleration of P in the direction PG

$$= m'/PQ^2 = m'^3/(m + m')^2\,PG^2,$$

and hence it follows that the path of P relative to G is a conic of which G is a focus.

This conic is a parabola, ellipse, or hyperbola according as u^2 is equal to, less than or greater than $2m'^3/(m + m')^2 AG$.

If the relative path is an ellipse, its transverse axis $2a$ is given by the equation

$$u^2 = \frac{2m'^3}{(m + m')^2}\left(\frac{1}{AG} - \frac{1}{2a}\right),$$

and the periodic time $= 2\pi a^{\frac{3}{2}}\,(m + m')/m'^{\frac{3}{2}}$.

In the same manner if the law of attraction be that of the inverse cube of the distance, the relative paths may be equiangular spirals, provided the relative velocities are properly adjusted.

172. If the two particles are not initially projected in the same plane, we must find the velocity of the centre of inertia of the system, and, by reversing it on each body of the system, we shall obtain their velocities relative to the centre of inertia. These velocities will be in parallel directions and such that $mu = m'u'$, and the plane passing through them is the plane of the relative motion.

Hence it is seen that the actual motion consists of the motion in the plane, while the plane, remaining parallel to itself, moves with the centre of inertia.

173. We can also obtain these results from the equations of motion.

For, if x, y, z be the co-ordinates of one particle, x', y', z' of the other, r the distance between them, and R the force of each on the other, the equations of motion are,

$$m\ddot{x} = -R\frac{x-x'}{r}, \quad m\ddot{y} = -R\frac{y-y'}{r}, \quad m\ddot{z} = -R\frac{z-z'}{r},$$

$$m'\ddot{x}' = R\frac{x-x'}{r}, \quad m'\ddot{y}' = R\frac{y-y'}{r}, \quad m'\ddot{z}' = R\frac{z-z'}{r}.$$

Hence we obtain

$$\frac{\ddot{x}-\ddot{x}'}{x-x'} = \frac{\ddot{y}-\ddot{y}'}{y-y'} = \frac{\ddot{z}-\ddot{z}'}{z-z'} = -R\frac{m+m'}{mm'r};$$

and, integrating, we find that

$$(y-y')(\dot{z}-\dot{z}') - (z-z')(\dot{y}-\dot{y}') = A,$$
$$(z-z')(\dot{x}-\dot{x}') - (x-x')(\dot{z}-\dot{z}') = B,$$
$$(x-x')(\dot{y}-\dot{y}') - (y-y')(\dot{x}-\dot{x}') = C,$$

A, B, and C being constants,

and $\therefore$ $\quad A(x-x') + B(y-y') + C(z-z') = 0,$

shewing that the line joining the particles is always perpendicular to the line

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C}.$$

174. *Motion of two heavy particles, connected by an inextensible string, and projected in any manner.*

If the initial distance of the particles from each other be less than the length of the string, there will be, after a certain time, a jerk of the string. The velocities perpendicular to the string will be unchanged, and the change of the velocities in direction of the string will be determined by the consideration that the momentum of the system in that direction will not be affected by the jerk.

For the subsequent motion, since the tension of the string produces equal and opposite momenta in any given time, it follows that the horizontal momentum, in any direction, of the system is constant, and that the vertical momentum imparted to the system is the same as if there were no string.

If then u, v be the component horizontal velocities in given directions, at any time, of one particle, and u', v' of the other,

$$mu + m'u' \text{ and } mv + m'v'$$

are each constant, and if w, w' be the initial vertical velocities, measured downwards, the vertical momentum of the system at the time t is

$$mw + m'w' + (m + m')\,gt.$$

Taking then α, β, γ as the component velocities of the centre of inertia, the horizontal resultant $\sqrt{\alpha^2 + \beta^2}$ is constant in magnitude and direction, and

$$(m + m')\,\gamma = mw + m'w' + (m + m')\,gt,$$

so that the motion of G is the same as that of a projectile.

Further, if we imagine the velocity and acceleration of G reversed on the system, we shall have the case of two particles connected with G by strings of given length, and consequently the motion of each will be circular, and the tension of the string will be constant. These are particular cases of the theorems of Arts. (41) and (45).

175. We can obtain the same result by means of the equations of motion.

If T be the tension and l the length of the string, these equations are

$$m\ddot{x} = -T\frac{x-x'}{l}, \quad m\ddot{y} = -T\frac{y-y'}{l}, \quad m\ddot{z} = mg - T\frac{z-z'}{l},$$

$$m'\ddot{x}' = T\frac{x-x'}{l}, \quad m'\ddot{y}' = T\frac{y-y'}{l}, \quad m'\ddot{z}' = mg + T\frac{z-z'}{l}.$$

The addition of the several pairs of equations gives the first result.

Also we find as in Art. (172) that the string is always perpendicular to a certain fixed direction.

Further we have

$$(x-x')^2 + (y-y')^2 + (z-z')^2 = l^2,$$

and therefore

$$(x-x')(\ddot{x}-\ddot{x}') + (\dot{x}-\dot{x}')^2 + \ldots\ldots\ldots = 0. \quad (\alpha).$$

But from the equations of motion

$$\ddot{x} - \ddot{x}' = -T\left(\frac{1}{m} + \frac{1}{m'}\right)\frac{x-x'}{l}, \text{ \&c.}$$

$$\therefore \quad (\dot{x}-\dot{x}')^2 + (\dot{y}-\dot{y}')^2 + (\dot{z}-\dot{z}')^2 = C - Tl\left(\frac{1}{m} + \frac{1}{m'}\right),$$

and substituting in the equation (α) we deduce that T is constant.

176. *Motion of a number of free particles, attracting each other with forces proportional to the distance.*

In this case, by a well-known theorem, the resulting action on each particle is directed to the centre of gravity of the system, and is proportional to the distance from it. The particles therefore describe, relative to G, ellipses of which G is the centre, and in the same periodic time.

EXAMPLES.

1. Two bodies attracting each other with a force which varies inversely as the cube of the distance are projected in parallel directions; find the condition that the relative paths may be equiangular spirals.

2. If two particles of masses μ, μ' attract according to the law of gravitation and be projected with velocities v, v', making an angle α with each other; shew that their orbits relative to their common centre of gravity will be parabolas, ellipses, or hyperbolas, according as

$$v^2 - 2vv' \cos\alpha + v'^2 \overset{=}{\underset{>}{<}} \frac{2(\mu+\mu')}{c},$$

where c is their initial distance apart.

3. Two bodies, the masses of which are m and m', are projected from the points A, B, and attract each other according to the Newtonian law. The body m is projected from A in the direction BA with a velocity $\sqrt{\dfrac{m+m'}{AB}}$, and m' is projected from B in a direction BP with a velocity

$$2\sqrt{\frac{m+m'}{AB}} \,.\, \cos PBA;$$

determine completely the path of either with regard to the other.

4. The co-ordinates (x, y), (x_1, y_1), of the simultaneous positions of two equal particles are given by the equations

$$x = a\theta - 2a\sin\theta, \qquad x_1 = a\theta,$$
$$y = a - a\cos\theta, \qquad y_1 = -a + a\cos\theta;$$

prove that, if they move under their mutual attractions, the law of force will be that of the inverse fifth power of the distance.

5. Two particles move under the action of their mutual attractions, one of them being constrained to remain on a fixed smooth wire in the form of a plane curve: if the path of the other be an involute to this curve and the two particles be always at corresponding points, the curve has for its intrinsic equation

$$s = a\epsilon^{-\frac{m\phi^2}{2}},$$

where m is the ratio of the masses of the particles.

6. Two equal bodies attract each other with a force varying inversely as the fifth power of the distance, and they are projected with equal velocities, in opposite directions, at right angles to the line joining them; prove that there are two velocities, in the ratio of $1 : \sqrt{2}$, for each of which the relative orbits will be circles.

7. Two masses m, m' are connected by an inextensible string of length a. The extremity A to which m is attached is compelled to move with uniform acceleration in a straight line under the action of a force P in the straight line, and the extremity B to which m' is attached, is compelled to describe a circle round A with uniform angular velocity ω under the action of a force Q perpendicular to AB. Find P and Q, and prove that the least value of P is

$$mf - \frac{m'a^2\omega^4}{4f} \text{ provided } a\omega^2 \text{ be } < 2f.$$

8. Two smooth circular rings (of radius a) are placed in a vertical plane with their centres in the same horizontal line at a distance $3a$. Two equal beads (of mass m) slide on these rings and are connected by a thin elastic string, of which the natural length is $3a$ and modulus of elasticity $3\lambda mg$. They are held as far apart as possible and then let go. Find when they come to rest.

In the particular case in which $\lambda = 1$, find the whole time of the motion.

9. Two equal particles can move on a fixed smooth circular wire and attract each other with a force varying as

the distance between them. Prove that their centre of gravity moves with uniform angular velocity, and that the relative motion of one with respect to the other is the same as the motion of a simple pendulum.

10. Two beads of equal mass repelling one another with a force varying inversely as the square of the distance are free to slide on a parabolic wire. If they are initially at the extremities of the latus rectum, prove that if properly projected the line joining them will always pass through the focus of the parabola.

11. The attraction between two equal particles, each of mass m, is $\mu m^2/r^3$, when r is the distance between them, and they are projected with equal velocities on the same side of the line (c) joining them in directions not parallel but equally inclined to that line; prove that the path of each will be an ellipse, parabola, or hyperbola, according as the initial component of each velocity in direction of the line c is less than, equal to, or greater than $\sqrt{2\mu m/c^2}$.

12. Two small rings, each of mass m, which attract each other with the force $m\omega^2 \times$ distance, are placed on smooth wires Ox, Oy, inclined to each other at a given angle, which commence to move in their own plane with angular velocity ω, and continue to move uniformly. Determine the motion of the rings.

CHAPTER XII.

ENERGY AND MOMENTUM.

177. It is intended in this Chapter to illustrate the use of the principles of momentum and energy which were laid down in Articles (40), (43) and (48) of Chapter IV.

In many cases problems of motion are very rapidly and easily solved by the aid of these principles, and in all the cases to which they are wholly or partially applicable, the problem of determining the motion of a body or a system is reduced to the solution of equations containing differential coefficients of the first order, or simple time-fluxes of the co-ordinates of the system.

178. *Motion of two spheres which attract each other according to the law of nature, of given masses m and m', and given radii a and a', placed originally without kinetic energy with their centres at a given distance c from each other.*

Supposing that the configuration of zero potential energy is when the spheres are in contact the potential energy of the initial configuration, which is the work done in separating the spheres,

$$=\int_{a+a'}^{c}\frac{mm'}{r^2}\,dr=\frac{mm'}{a+a'}-\frac{mm'}{c}.$$

During the subsequent motion let u and u' be the velocities of the two balls when their centres are at a distance r from each other.

The principles of momentum and energy give the two equations,

$$mu = m'u',$$

$$\tfrac{1}{2}mu^2 + \tfrac{1}{2}m'u'^2 + \frac{mm'}{a+a'} - \frac{mm'}{r} = \frac{mm'}{a+a'} - \frac{mm'}{c},$$

or
$$\tfrac{1}{2}(mu^2 + m'u'^2) = mm'\left(\frac{1}{r} - \frac{1}{c}\right),$$

from which u and u' are at once determined in terms of the distance.

179. *Motion of a simple pendulum.*

If a simple pendulum of length l start from the inclination α to the vertical, the work done by gravity as the pendulum falls to the inclination θ is $mg(l\cos\theta - l\cos\alpha)$, and the kinetic energy is $\tfrac{1}{2}ml^2\dot{\theta}^2$.

Equating these we find that

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\alpha),$$

$$\text{and } \therefore \frac{dt}{d\theta} = \sqrt{\frac{l}{2g}}\,\frac{1}{\sqrt{\cos\theta - \cos\alpha}}.$$

If α is very small, this is approximately

$$\frac{dt}{d\theta} = \sqrt{\frac{l}{g}}\,\frac{1}{\sqrt{\alpha^2 - \theta^2}},$$

from which we obtain $\theta = \alpha\cos\sqrt{\frac{g}{l}}\,t$ as in Art. (126).

180. *Motion of two equal heavy particles, fastened to the ends of a rod without weight, and oscillating in a vertical plane inside a smooth sphere.*

Taking a for the radius and $2c$ for the length of the rod, the equation of energy is

$$2\left\{\tfrac{1}{2}ma^2\dot{\theta}^2\right\} = 2mg\sqrt{a^2 - c^2}\,(\cos\theta - \cos\alpha),$$

θ being the inclination of the rod to the horizon,

or
$$\dot{\theta}^2 = \frac{2g\sqrt{a^2-c^2}}{a^2}(\cos\theta - \cos\alpha).$$

Comparing this with the first equation of the last article we see that the length of the equivalent simple pendulum is

$$a^2 \div \sqrt{a^2-c^2}.$$

181. *Motion of a compound pendulum, that is, of any rigid body, or connected system of bodies, about a fixed horizontal axis.*

G being the centre of gravity of the system and GO its distance from the axis, let θ be the inclination of GO to the vertical at any time during the motion.

If P be the position of a particle mass m of the system, and if $OP = r$, the kinetic energy of the particle m is $\frac{1}{2}mr^2\dot{\theta}^2$, for the angular velocity of OP is the same as that of OG.

Hence the equation of energy gives

$$\Sigma\left(\tfrac{1}{2}mr^2\dot{\theta}^2\right) = Mga(\cos\theta - \cos\alpha),$$

if $OG = a$, and $M =$ the total mass.

The expression $\Sigma(mr^2)$ is called the moment of inertia of the system about the axis, and is generally represented by the expression Mk^2, so that

$$\dot{\theta}^2 = \frac{2ag}{k^2}(\cos\theta - \cos\alpha).$$

Comparing this with the first equation of Art. (179), we see that the length of the equivalent pendulum is $k^2 \div a$, so that if l be the length,

$$la = k^2.$$

The point in OG at the distance l from O is called the centre of oscillation of the system, the point O being the centre of suspension.

182. The evaluation of the expression $\Sigma(mr^2)$ for particular cases is an exercise in the Integral Calculus.

In the performance of the calculation two facts are of great utility; these are

(1) That the moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity together with the product of the mass by the square of the distance between those parallel axes;

(2) That the moment of inertia of a plane lamina about any axis perpendicular to the plane is equal to the sum of the moments of inertia about any two axes in the plane, perpendicular to each other, drawn through the foot of the axis.

For the first let r' be the distance of any particle from the axis, and r its distance from the parallel axis through G.

Then if h be the distance between the axes,

$$\Sigma(mr'^2) = \Sigma m(r^2 + h^2 - 2hr\cos\theta) = \Sigma(mr^2) + Mh^2.$$

For the second, if x and y be the distances of any particle of the lamina from the axes in a plane

$$\Sigma(mr^2) = \Sigma m(x^2 + y^2) = \Sigma(mx^2) + \Sigma(my^2).$$

For convenience a few simple results may be stated, taking in all cases M as the mass of the body, and Mk^2 as representing the moment of inertia.

For a straight rod of length l about an axis through one end perpendicular to the rod, $k^2 = \frac{1}{3}l^2$; and hence, if the axis pass through the middle point, $k^2 = \frac{1}{12}l^2$.

For a circular ring, about the line through its centre perpendicular to its plane, $k^2 = r^2$. For a circular lamina about the line through its centre perpendicular to its plane $k^2 = \frac{1}{2}r^2$

For a parallelopiped of edges a, b, c, about the edge c,

$$k^2 = \frac{1}{3}(a^2 + b^2).$$

For an elliptic area about the transverse and conjugate axes, $k^2 = \frac{1}{4}b^2$, and $k^2 = \frac{1}{4}a^2$,

and about the line through its centre perpendicular to its plane

$$(k^2 = \tfrac{1}{4}a^2 + b^2).$$

For a sphere about a diameter, $k^2 = \frac{2}{5}r^2$.

For a solid ellipsoid about the axis c,

$$k^2 = \tfrac{1}{5}(a^2 + b^2).$$

183. *Expressions for linear and angular momenta.*

Considering motion in one plane, if x and y be the co-ordinates of a particle m of the system, the linear momenta parallel to the axes, are $\Sigma(m\dot{x})$ and $\Sigma(m\dot{y})$, or, if ξ, η be the co-ordinates of the centre of gravity, $M\dot{\xi}$ and $M\dot{\eta}$.

The moments about the origin of the momenta $m\dot{x}$ and $m\dot{y}$ being $-m\dot{x}y$ and $my\dot{x}$, the angular momentum of the system is

$$\Sigma m(x\dot{y} - y\dot{x}).$$

In polar co-ordinates $mr\dot{\theta}$ is the part of the momentum perpendicular to the radius vector, and therefore the angular momentum is

$$\Sigma(mr^2\dot{\theta}).$$

If A be the area swept over by the radius vector,

$$2\dot{A} = x\dot{y} - y\dot{x} = r^2\dot{\theta},$$

so that the angular momentum is

$$2\Sigma m\dot{A}.$$

Again, if p be the perpendicular on the tangent to the path of the particle m, the angular momentum of the system is

$$\Sigma(m\dot{s}p),$$

and, since $pds = r^2 d\theta$, and $p = x\dfrac{dy}{ds} - y\dfrac{dx}{ds}$,

this expression at once transforms itself into either of those preceding.

In the case of a single rigid body, revolving about an axis with which it is rigidly connected, $\dot{\theta}$ is the same for all the particles, and the angular momentum is

$$Mk^2\dot{\theta}, \text{ or } Mk^2\omega.$$

184. For motion in three dimensions the expressions for the linear and angular momenta are

$$\Sigma (m\dot{x}),\ \Sigma (m\dot{y}),\ \Sigma (m\dot{z}),$$
$$\Sigma m (y\dot{z} - z\dot{y}),\ \Sigma m (z\dot{x} - x\dot{z}),\ \Sigma m (x\dot{y} - y\dot{x}).$$

If we take ξ, η, ζ as the co-ordinates of the centre of gravity, and x, y, z as the co-ordinates, relative to G, of a particle m, the angular momentum about the axis of z

$$= \Sigma \{m (\xi + x) (\dot{\eta} + \dot{y}) - (\eta + y) (\dot{\xi} + \dot{x})\},$$
$$= M (\xi\dot{\eta} - \eta\dot{\xi}) + \Sigma m (x\dot{y} - y\dot{x}),$$

since $\Sigma (mx) = 0$, and $\Sigma (my) = 0$.

Hence it follows that the angular momentum of a system, about any assigned axis, is the sum of the angular momentum due to the motion of the particles of the system relative to the parallel axis through the centre of gravity, and of the angular momentum due to the mass of the system supposed to be concentrated at, and moving with, the centre of gravity.

It may be instructive to present the proof of this statement in another form.

Let O and G be the projections of the assigned axis and of the centre of gravity on a plane perpendicular to the axis, GA the projection of the line of motion of G, and FK the projection of the line of motion of a particle.

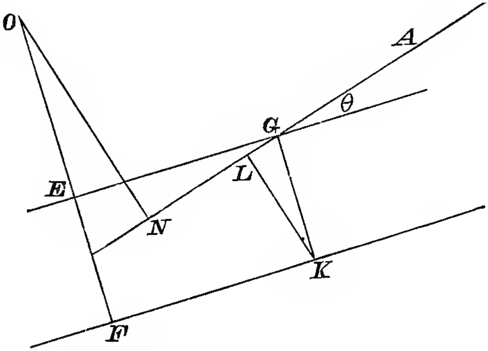
Taking m as the mass of the particle and u as the component in the line FK of its velocity, and taking EG parallel to FK,

The angular momentum about the axis O

$$= \Sigma (mu\,.\,OF) = \Sigma (mu\,.\,OE) + \Sigma (mu\,.\,GK).$$

If $\bar{u}$ be the velocity of G in the direction GA, the first term, which is the sum of the moments about O of a number of momenta in lines through G, is equal to the moment of their resultant, and therefore

$$= M\bar{u} \, . \, ON.$$



The second term

$$= \Sigma \, \{m \, (u - \bar{u} \cos \theta) \, GK\} + \bar{u} \Sigma \, (m \, GK \cos \theta).$$

But $$\Sigma \, (m \, GK \cos \theta) = \Sigma \, (m \, . \, KL) = 0,$$

and therefore the second term represents the angular momentum due to the motion of the particles of the system relative to the centre of gravity.

185. In the case of a single rigid body, when the motion is entirely in two dimensions, an expression for the angular momentum is

$$M\rho^2 \dot{\phi} + Mk^2 \omega,$$

if ρ and ϕ be the co-ordinates of the centre of gravity.

Take, for instance, the case of a number of rigid bodies, rotating about the same fixed axis with given angular velocities, and becoming suddenly, or gradually, connected together.

In this case the total angular momentum is unchanged, and therefore, if $mk^2\omega$ be the original angular momentum of one of the bodies, and Ω the final angular velocity of the system,

$$\Omega \,.\, \Sigma\,(mk^2) = \Sigma\,(mk^2\omega).$$

186. *Expressions for kinetic energy.*

For motion in two dimensions, the expression in rectangular co-ordinates is

$$\tfrac{1}{2}\Sigma m\,(\dot{x}^2 + \dot{y}^2),$$

and in polar co-ordinates,

$$\tfrac{1}{2}\Sigma m\,(\dot{r}^2 + r^2\dot{\theta}^2).$$

For the case of a rigid body in motion about a fixed axis, the kinetic energy is

$$\tfrac{1}{2}\Sigma\,(mr^2\dot{\theta}^2) \text{ or } \tfrac{1}{2}Mk^2\dot{\theta}^2.$$

For motion in three dimensions the expression for the kinetic energy in rectangular co-ordinates, is

$$\tfrac{1}{2}\Sigma m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2);$$

in cylindrical co-ordinates,

$$\tfrac{1}{2}\Sigma m\,(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2);$$

and in polar co-ordinates,

$$\tfrac{1}{2}\Sigma m\,(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2).$$

Other modes of expression can also be given and will be employed when necessary.

If x, y, z be co-ordinates relative to the centre of gravity the kinetic energy

$$= \tfrac{1}{2}\,.\,\Sigma\,\{m\,(\dot{\xi} + \dot{x})^2 + (\dot{\eta} + \dot{y})^2 + (\dot{\zeta} + \dot{z})^2\},$$

$$= \tfrac{1}{2}\mathrm{M}\,(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) + \tfrac{1}{2}\Sigma m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

So that the kinetic energy is the sum of the energy due to a mass *M* at the centre of gravity and of the energy due to the motions of the particles of the system relative to the centre of gravity.

187. In all cases in which no external forces are in action the linear momenta and the angular momenta remain constant.

The principle of the conservation of energy of a mechanical system, that is to say, the assertion that the sum of the potential and of the visible kinetic energies is constant, applies to all those cases in which the potential energy depends on the configuration of the system, and in which the change of potential energy, due to a change of configuration, is independent of the manner in which that change is made.

There is no doubt that, in any system, the total energy remains unchanged, unless extraneous force act on the system, but, as in the case of impacts taking place between bodies of the system, there may be an apparent loss of kinetic energy, which is fully accounted for by the development of invisible kinetic energy in the form of heat, or in some other form.

Or again, kinetic energy may be created by explosions, but in this case, the visible kinetic energy, together with the heat added to the system, are the equivalent of the potential energy which was lying dormant in the explosive matter while in its quiescent condition.

Internal friction, if due to the sliding of two surfaces on each other, is destructive of visible kinetic energy, while, on the other hand, visible kinetic energy may be created by the action of live things, as for instance in the case of a man walking on a rough plank, or a rough ball, or climbing a moveable rope.

In such cases the system is said to be not dynamically conservative, although it is really conservative if all the transformations of energy be taken into the account.

188. If an elastic string form part of a system, the gain of potential energy due to its extension from its natural length is

$$\tfrac{1}{2}\,(\text{Tension})\,(\text{Extension});$$

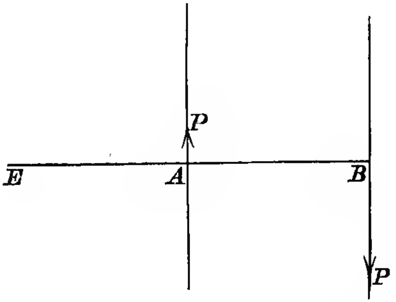
for the work done in pulling out the string

$$=\int_l^r T dx = \int_l^r \frac{\lambda}{l}(x-l)\,dx = \tfrac{1}{2}\frac{\lambda}{l}(r-l)^2,$$

as may also be shewn by a simple geometrical figure. In such a case the contraction or extension of the string implies a transformation of energy, the loss, or gain, of potential energy due to the contraction or extension, being represented by a change in either, or both, of the kinetic energy, and of the potential energy due to configuration, irrespective of the string.

Energy imparted to a system by the action of an extraneous couple.

Any state of motion of a plane can be represented by a state of rotation about an instantaneous centre in the plane.



If a small displacement, $\delta\phi$, be made round the instantaneous centre E, the work done by the force of the couple

$$= P \,.\, EB\delta\phi - P \,.\, EA\delta\phi = P \,.\, AB\delta\phi = G\delta\phi,$$

G being the moment of the couple.

Hence, if θ be the total angular twist of the couple, the work done

$$= \int_0^\theta G d\phi = G\theta, \text{ if } G \text{ be constant.}$$

We now proceed to illustrate these general statements by their applications to some particular cases.

189. *Motion of a heavy rod placed with its lower end in contact with a smooth horizontal plane, and let go.*

If α be the initial inclination to the vertical, and θ at any subsequent time, the work done by gravity is

$$mg\,a\,(\cos\alpha - \cos\theta),$$

and the kinetic energy is

$$\tfrac{1}{2}m\,(\dot{y}^2 + k^2\dot{\theta}^2), \text{ where } y = a\cos\theta.$$

Hence we obtain

$$\frac{a^2}{2}(\sin^2\theta + \tfrac{1}{3})\,\theta^2 = ag\,(\cos\alpha - \cos\theta),$$

or

$$\dot{\theta}^2 = \frac{6g}{a}\,\frac{\cos\alpha - \cos\theta}{3\sin^2\theta + 1}.$$

190. *Motion of a heavy rod, constrained to slide in a vertical line, with its lower end on the curved surface of a smooth hemisphere, the hemisphere sliding on a smooth horizontal plane.*

If θ be the inclination to the vertical of the radius to the point of contact, and α its initial value, the work done by gravity is $mga\,(\cos\alpha - \cos\theta)$, m being the mass of the rod. If M be the mass of the hemisphere the kinetic energy of the system is

$$\tfrac{1}{2}m\left(\frac{d}{dt}\,.\,a\cos\theta\right)^2 + \tfrac{1}{2}M\left(\frac{d}{dt}\,a\sin\theta\right)^2,$$

and therefore

$$(m\sin^2\theta + M\cos^2\theta)\,\theta^2 = \frac{2mg}{a}(\cos\alpha - \cos\theta).$$

191. *Motion of a heterogeneous sphere rocking on a rough horizontal plane, the whole motion being parallel to one vertical plane.*

Taking O as the point of contact of the end of the radius CG through G when it is vertical, x, y, as the co-ordinates of G, $CG = c$, and the radius $= a$, the equation of energy is

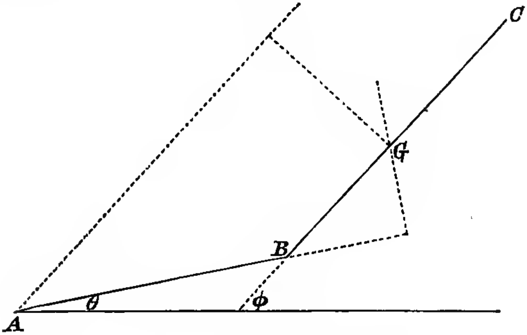
$$\tfrac{1}{2}m\,\{\dot{x}^2 + \dot{y}^2 + k^2\dot{\theta}^2\} = mgc\,(\cos\theta - \cos\alpha),$$

where $x = a\theta - c\sin\theta$, and $y = a - c\cos\theta$;

$$\therefore\quad \dot{\theta}^2\,\{a^2 + c^2 - 2ac\cos\theta + k^2\} = 2gc\,(\cos\theta - \cos\alpha).$$

192. *Motion of two equal rods, AB, BC, jointed at B and moving, on a smooth horizontal plane, about the fixed point A.*

In this case the angular momentum about A and the kinetic energy are both constant.



Taking θ and ϕ as the inclinations to a fixed line on the plane, the velocity of the centre of gravity, G, of BC is compounded of its velocity relative to B,
that is, $a\dot\phi$ perpendicular to BC, and of the velocity of B, which is $2a\dot\theta$ perpendicular to AB;
the angular momentum of AB is $m\frac{4a^2}{3}\dot\theta$, and that of BC is

$$m\frac{a^2}{3}\dot\phi + ma\dot\phi\{a + 2a\cos(\phi-\theta)\} + m2a\dot\theta\{2a + a\cos(\phi-\theta)\};$$

hence the equation of momentum is

$$\dot\theta\left(\tfrac{16}{3} + 2\cos\overline{\phi-\theta}\right) + \dot\phi\left(\tfrac{4}{3} + 2\cos\overline{\phi-\theta}\right) = C.$$

Again, the square of the velocity of G

$$= a^2\dot\phi^2 + 4a^2\dot\theta^2 + 4a^2\dot\phi\dot\theta\cos\overline{\phi-\theta},$$

the kinetic energy of AB is $m\frac{4a^2}{3}\dot\theta^2$,

and that of BC due to rotation is $m\frac{a^2}{3}\dot\phi^2$;

∴ the equation of energy is

$$\tfrac{16}{3}\,\dot\theta^2 + \tfrac{4}{3}\,\dot\phi^2 + 4\dot\phi\dot\theta\cos\overline{\phi-\theta} = C',$$

C and C' being constants given by the initial conditions.

These two equations determine $\dot\theta$ and $\dot\phi$.

193. If a solid body is rotating about a fixed axis, and is changing its shape and size, without the action of any external force, its angular momentum remains constant, and this determines the change of angular velocity.

If for instance a sphere rotate about a fixed diameter, changing its size, but retaining its shape and its homogeneity, the angular momentum, which is $\frac{2}{5}Mr^2\omega$, remains constant.

If the radius change to r', the change in the kinetic energy is

$$\tfrac{1}{5}M\left(r'^2\omega'^2 - r^2\omega^2\right) \text{ or } \tfrac{1}{5}M\frac{r^2}{r'^2}\left(r^2 - r'^2\right)\omega^2.$$

194. *Motion of a heavy rod swinging about its upper extremity which is freely jointed to a fixed support.*

If θ be the inclination of the rod to the vertical, and $\dot\phi$ its azimuthal velocity, the angular momentum about the vertical through the fixed end and the energy are respectively,

$$\Sigma\left\{mr^2\sin^2\theta\,.\,\dot\phi\right\}$$

and $$\tfrac{1}{2}\Sigma\left\{mr^2\dot\theta^2 + mr^2\sin^2\theta\,.\,\dot\phi^2\right\} + Mg\left(a - a\cos\theta\right),$$

if we assume that the configuration for zero energy is when the rod is vertical and at rest.

Each of these expressions being constant, we obtain for the determination of θ and ϕ the equations,

$$\dot\phi\sin^2\theta = \Omega\sin^2\alpha,$$

$$\tfrac{2}{3}a\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + g\left(1-\cos\theta\right)$$
$$= \tfrac{2}{3}a\left(\omega^2 + \Omega^2\sin^2\alpha\right) + g\left(1-\cos\alpha\right),$$

ω and Ω being the initial values of $\dot\theta$ and $\dot\phi$.

Eliminating $\dot{\phi}$ and taking the time-flux of the resulting equation, we obtain

$$4a\ddot{\theta} - 4a\Omega^2 \sin^4 \alpha \operatorname{cosec}^2\theta \cot\theta + 3g \sin\theta = 0.$$

The condition that the rod may revolve uniformly, so as to describe a right circular cone of vertical angle 2α, is

$$4a\Omega^2 \cos\alpha = 3g.$$

If while thus revolving the rod receive a slight displacement, not disturbing this relation, the small vibrations in θ will be determined by putting $\theta = \alpha + \psi$, when ψ is small in the above equation.

The result is the approximate equation,

$$4a \cos\alpha \ddot{\psi} + 3g (1 + 3 \cos^2\alpha) \psi = 0,$$

so that the period of the oscillation is

$$4\pi \sqrt{a \cos\alpha} / \sqrt{3g (1 + 3 \cos^2\alpha)}.$$

195. *Motion of a heavy horizontal ring, fitting on a smooth vertical cylinder, and supported by a number of vertical strings fastened to its upper edge.*

If the ring, when in equilibrium, be started with a given angular velocity ω, it will rise until the potential energy stored up by the elevation is equal to the original kinetic energy, that is to a height h such that

$$mgh = \tfrac{1}{2} ma^2\omega^2,$$

a being the radius.

As the ring rises the strings will form helices on the surface of the cylinder, and, taking l for the length of each string, and θ the angle through which the ring turns in rising through the height z, we have the geometrical equation

$$l^2 = (l - z)^2 + a^2\theta^2, \text{ and } \therefore a^2\theta\dot{\theta} = (l - z)\dot{z}.$$

The equation of energy is

$$\tfrac{1}{2} m (\dot{z}^2 + a^2\dot{\theta}^2) = \tfrac{1}{2} ma^2\omega^2 - mgz,$$

and these equations give $\dot{\theta}$ and $\dot{z}$ in terms of z.

If, instead of starting the ring with an angular velocity, it be set in motion by a horizontal couple G, twisted through an angle β, the energy imparted to the system is $G\beta$.

Part of this energy appears in the form of a change of potential energy due to change of configuration, and its amount is

$$mg\{l-\sqrt{l^2-a^2\beta^2}\}.$$

The difference, $G\beta-mg\{l-\sqrt{l^2-a^2\beta^2}\}$, is represented by the kinetic energy with which the ring starts off in its new configuration.

196. *If a heavy elastic ring, in the form of a horizontal circle, be placed on the surface of a smooth sphere and allowed to slip down,* we can determine the motion by observing that the kinetic energy is increased by the fall, and diminished by the extension of the ring. If a be the radius of the sphere, $2\pi a\sin\alpha$ the initial length of the ring, and θ be the angular distance of each point of the ring from the vertex at any subsequent time, the equation of energy is

$$\tfrac{1}{2}ma^2\dot{\theta}^2=mga(\cos\alpha-\cos\theta)-\frac{\lambda}{2}\,.\,2\pi a\frac{(\sin\theta-\sin\alpha)^2}{\sin\alpha}.$$

The condition that the string should just slip over the sphere is that

$$\dot{\theta}=0 \text{ when } \theta=\frac{\pi}{2},$$

which gives $$mg\sin\alpha\cos\alpha=\pi\lambda(1-\sin\alpha)^2.$$

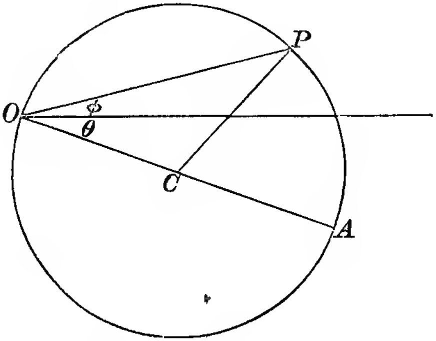
197. *A circular wire ring of mass M, carrying a small bead of mass m, lies on a smooth horizontal table, and is capable of turning about a fixed point in its circumference. An elastic thread the natural length of which is less than the diameter of the ring has one end fastened to the bead and the other end to the fixed point. It is required to determine the motion when the initial position of the thread coincides very nearly with the diameter of the ring.*

The angular momentum is always zero, so that if OP the thread be inclined at an angle ϕ to the initial position of the diameter OA,

$$M\,.\,2a^2\dot{\theta} = mr^2\dot{\phi},$$

where

$$r = 2a\cos(\phi + \theta).$$



Again the equation of energy is

$$M\,.\,2a^2\dot{\theta}^2 + m(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{\lambda}{l}\{(2a-l)^2 - (r-l)^2\},$$

and these equations theoretically determine θ and ϕ.

When the string contracts to its natural length the kinetic energy retains a constant value until by the motion of the bead the string is again put into a state of tension.

198. *If an elastic thread, in the form of a circle on a smooth horizontal plane, be set rotating with a given angular velocity,* the principles of angular momentum and energy give the equations

$$mr\dot{\theta}^2 = ma^2\omega,$$

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \tfrac{1}{2}ma^2\omega^2 - \frac{\lambda\pi}{2a}(r-a)^2;$$

and those equations determine $\dot{r}$ and $\dot{\theta}$ in terms of r.

199. *Motion of a wire ring on a smooth horizontal plane produced by an insect alighting upon it and moving uniformly round the arc of the ring.*

As the insect starts with a finite motion there must be an impulsive action, that is, of the nature of a kick, between the insect and the ring in direction of the tangent.

Take M, m, as the masses and u as the velocity of the insect relative to the ring.

If V, v be the actual initial velocities, in contrary directions, of the centre of the ring and of the insect,

$$MV = mv.$$

If ω be the initial angular velocity of the ring, we have the geometrical condition,

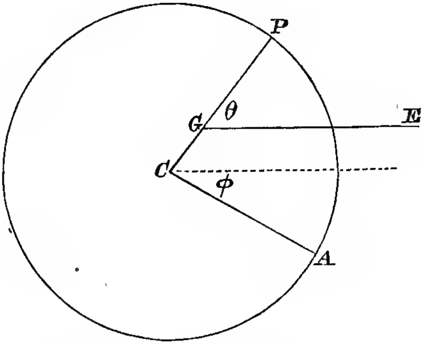
$$V + a\omega + v = u.$$

Again the angular momentum about any vertical line is equal to zero.

Taking the angular momentum about the vertical line through the initial position of the centre of the ring,

$$Ma^2\omega = mva,$$

and these equations determine V, ω and v.



During the subsequent motion, G remains at rest, and the angular momentum about G is zero, so that if ϕ and θ be the angles turned through by the ring and the radius to the insect,

$$mPG^2\dot{\theta} + MCG^2\dot{\theta} - Ma^2\dot{\phi} = 0,$$

or

$$m\dot{\theta} = (M + m)\dot{\phi}.$$

This equation, and the geometrical condition, $a(\dot{\theta} + \dot{\phi}) = u$, determine θ and ϕ.

200. A remarkable application of the principle of energy has been made by Professor C. Niven in discussing the motion produced when a heavy elastic string, suspended from one end, is cut at any point*.

To illustrate this application we take a simple case, in which the action of gravity has no effect.

An elastic string has one end fastened to a fixed point on a smooth horizontal plane, and the other end is drawn out to any assigned distance, and is then let go; it is required to determine the subsequent motion.

Taking a as the natural length and l as the stretched length of the string, the tension T is $\lambda(l-a)/a$, and if dx be the natural length of an element at the end of the string, its potential energy, before the end is let go, which is half tension × extension,

$$= \tfrac{1}{2}\frac{T^2}{\lambda}\,dx = \tfrac{1}{2}.\lambda\left(\frac{l-a}{a}\right)^2 dx.$$

When the end is let go, this energy is set free, and is immediately converted into kinetic energy.

Taking m as the mass of unit length, and V as the instantaneous velocity of the end of the string, the kinetic energy is $\tfrac{1}{2}m\,dx\,V^2$, and therefore,

$$V = \frac{T}{\sqrt{m\lambda}} = \sqrt{\frac{\lambda}{m}}\,\frac{l-a}{a}.$$

The first element being started with this velocity the next element will be under the same initial conditions, and will acquire the same velocity, and all the elements in succession will acquire the same velocity, so that a wave like motion will be propagated from the free end to the fixed end of the string.

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* On a case of Wave motion, by Professor C. Niven, *Messenger of Mathematics.* Vol. VIII.

To find the velocity of transmission of the wave, we observe that when the wave is passing over the element PQ,

the tension at P is T and the tension at Q is zero, and therefore in the time dt the momentum generated in PQ is Tdt.

But this momentum is $mdx . V$, and therefore

$$\dot{x} = T/mV = \sqrt{\frac{\lambda}{m}},$$

so that the velocity of transmission is constant with regard to the natural length.

Hence the time in which the wave travels from B to $A = a\sqrt{m}/\sqrt{\lambda}$, and in this time the free end B will have traversed the space $Va\sqrt{m}/\sqrt{\lambda}$ which is equal to $l - a$.

It follows therefore that, when the wave reaches A, the string is in its unstretched condition, and every element has the same velocity V in the direction BA.

If we assume that Hook's Law holds for compression as well as for dilatation, the kinetic energy will be gradually converted into the potential energy of compression, which will in its turn be reconverted into the original kinetic energy.

If Hook's law does not hold for compression or if the string is so constituted that it is reduced to rest on arriving at its natural condition, the kinetic energy, which is apparently lost, is really converted into heat.

201. We can explain from first principles why it is that the release of the end of the string instantaneously results in the production of a finite velocity at that end.

Considering a small element at the end B, at the moment of release, the force on the element is T and its mass is mdx, so that we have a finite force acting on an infinitesimal mass.

Now if, by means of a mental microscope, we imagine the mass magnified so as to become a finite mass, and the force T magnified in the same proportion, we shall have the case of a very large force acting upon a finite mass, the result of which is, as we know, the production of a finite velocity in a very small time, and the smaller we take the element at the end B, the more closely do we tend to the ultimate form, which is the instantaneous production of velocity at the end.

EXAMPLES.

1. A heavy particle slides down a chord of a vertical circle to the lowest point of the circle, and then ascends the curve; prove that in no case can it rise through more than one-sixth of the total arc.

2. A and B are two pegs very near together and in the same horizontal line. A perfectly flexible string of length l is fastened at A and hangs over B, the portion between A and B hanging down; the loose end begins to descend from the position of equilibrium: prove that the final velocity of the string is $\sqrt{2gl/3}$.

3. If a system of bodies initially at rest be acted on by no forces but the attractions of its several parts, and at any subsequent time become united as a single solid body, shew that this body will be at rest.

4. A particle is attached to a smooth string which passes over a rough circular arc in a vertical plane; the particle, initially at the end of a horizontal diameter, is drawn up with constant acceleration g/π; prove that the work expended in drawing it to the vertex of the circle is to the work which would be done in lifting it through the same height in the ratio $$6 + 4\mu - \pi\mu \;:\; 4.$$

5. The two ends of a homogeneous rod are moveable on the arc of a fixed smooth conic, having its axis vertical and vertex downwards; if the length of the rod be greater than the latus-rectum, and if it be placed in a horizontal position and slightly displaced, find the greatest kinetic energy which it acquires.

6. A small heavy ring, moveable on a smooth circular wire fixed in a vertical plane, is attached by an elastic string without weight to a point at a distance below the lowest point of the circle equal to its radius, and when the ring is at the lowest point of the circle the string is unstretched. If the ring be drawn along the wire till the string becomes a tangent to the circle and then let go, determine the velocity with which it will reach the lowest point of the circle.

7. A particle is placed in an uniform straight tube, which is moveable in a horizontal plane about an axis through one end, and the mass of which is equal to that of the particle. The tube being set in motion, shew that the angle, which the direction of motion of the particle makes with the axis of the tube on leaving it, lies between $\frac{\pi}{2}$ and $\tan^{-1}\frac{1}{2}$.

8. A cylinder whose surface is smooth, stands on a smooth horizontal plane on one of its circular ends; find the impulse at a given point of its surface which will cause it to fall over.

9. The nut of a screw rests on a smooth horizontal plane, over a hole cut so as to allow a free passage for the screw, and the screw descends through the nut by its own weight; determine the motion, 1st, when the nut is fixed, 2nd, when it is moveable.

10. Two equal particles, connected by a stiff wire without weight, are placed in a vertical circular tube, one being at the highest point, shew that, when the other reaches the lowest point, the velocity of each is the same as if they had been unconnected from the beginning of the motion.

11. A ring slides on a smooth wire bent into the form of a plane vertical curve and is attached by an elastic string to a fixed point in the plane of the curve; if it start initially from a position in which the string is just not stretched, prove that it will descend through a vertical space which is a third proportional to the natural length of the string and its extension at the lowest position, supposing that the modulus of elasticity is twice the weight of the ring, and the string is stretched throughout the motion.

12. Two equal weights W are connected by a string of length $2l$, whose weight per unit of length is w, which passes over a small pully. The system is put in motion by adding a weight W' at one end. Shew that when either weight has moved through a distance x, the kinetic energy will be greater than if the string were weightless by $\frac{1}{2}(l-x)^2 w$.

13. The extremities of a uniform heavy rod of length $4a$ slide on the circumference of a three-cusped hypocycloid whose plane is vertical, one of the cusps being at the highest point of the circumscribing circle whose radius is $3a$; prove that the length of the isochronous simple pendulum is $4a/3$.

14. Four equal particles connected by inelastic strings, forming the sides of a square, mutually repel each other with a force μ (distance); if one string be cut, then (θ) the angle either string makes with its original position is given by

$$\dot{\theta}^2 . (2-\sin^2\theta) = 4\mu \sin\theta\,(2+\sin\theta).$$

15. A number of uniformly distributed particles move with the same velocity v in the same direction. In this medium is placed a body of any form and such that all the particles impinging on it adhere. Shew that, if M denote the mass of the body at any time, and u its velocity $M(v-u)$ will be constant. What information can be drawn from this fact as to the ultimate state of motion, and the time when this is arrived at?

16. Two equal shells are connected by a thin tube, the whole system having no mass, and they contain matter which is transferred uniformly from one to the other in time t

When the masses in each are equal the system has an angular velocity ω communicated to it. Shew that the curve traced out by the middle point of the tube is given by the polar equation $r = a \tanh 2\theta/\omega t$, where $2a$ is the distance between the centres of the shells.

17. A particle of given weight is fastened to a string which passes over a smooth circular arc in a vertical plane: the particle initially at the extremity of a horizontal diameter is drawn up to the vertex with constant acceleration g: prove that the work done in drawing it up the last half is to the work done in drawing it up the first half as

$$4\sqrt{2} + \pi - 4 : 4 + \pi\sqrt{2}.$$

18. Two equal particles are revolving in the same direction in the same ellipse, under the action of a force tending to a focus; shew that, if they become rigidly connected when they are at the extremities of a focal chord, they will afterwards move about their centre of gravity with an angular velocity which varies inversely as the length of the chord, and that, wherever this takes place, the initial velocity of the centre of gravity will be the same.

19. A heavy elastic string, in form of a ring, is placed, with its plane horizontal, over a smooth cone, the axis of which is vertical; find the position of equilibrium. Also, if the string be held in contact with the cone at its natural length, and let go, find how far it will descend.

20. Three equal particles at rest, tied together by three equal strings of length a, so as to form an equilateral triangle, repel each other with a constant force f. If one string be cut, shew that the equation to determine the motion is

$$a\dot{\theta}^2 (3 - 2\sin^2\theta) = 3f(2\sin\theta - 1),$$

2θ being the angle between the strings.

21. A uniform rod is placed with one extremity at the middle point of the line joining two equal centres of force attracting inversely as the square of the distance, and at right angles to that line; find the velocity with which the centre of the rod will reach the line.

22. Two equal inelastic rods of given length, fastened together by a smooth hinge, are placed on a vertical plane with their other extremities at opposite sides of a hole of given size in a smooth horizontal table. If motion be allowed to take place, determine the condition that the rods may just reach their position of unstable equilibrium.

23. A cube is rotating with angular velocity ω about a diagonal, when one of its edges which does not meet that diagonal becomes fixed; shew that the angular velocity about this axis will be $\omega/4\sqrt{3}$.

24. One end of a string (length a) is attached to a smooth circular wire of radius a, whose plane is vertical, at one extremity of its horizontal diameter, and the other end to one extremity of an inelastic rod of length a, the other extremity of which is made to slide along the wire by means of a small ring. If the rod and string are held initially in a horizontal position, and then abandoned to the action of gravity, the rod will first come to rest when its middle point is at a distance $\frac{13\sqrt{3}}{80}a$ below the horizontal diameter.

25. *ABCDE* is a pentagon, formed of five equal uniform rods freely jointed together. The pentagon is placed with its plane vertical, and angular point *A* uppermost; the angles *B* and *E* slide freely along a smooth horizontal rod. Investigate a differential equation of the second order for the determination of the inclination of *BC* or *DE* to the vertical.

26. A ring rests upon two smooth horizontal bars which in the position of equilibrium subtend an angle $2a$ at the centre; shew that, if the ring be disturbed by twisting i through a small angle about its vertical diameter, the length of the simple isochronous pendulum will be $\frac{c}{2}\cot a \operatorname{cosec} a$.

27. A fine circular tube, radius c, lies on a smooth horizontal plane, and contains two equal particles connected b an elastic string in the tube, the natural length of which i equal to half the circumference. The particles are in contac

and fastened together, the string being stretched round the tube.

If the particles become disunited, prove that the velocity of the tube when the string has regained its natural length is

$$\sqrt{\frac{2\pi\lambda mc}{M(M+2m)}},$$

where M, m are the masses of the tube and each particle, and λ is the modulus of elasticity.

28. If in the previous question one of the particles be fixed in the tube, and if the tube be moveable about the point at which the particle is fixed, prove that when the string has regained its natural length the angular velocity of the tube

$$=\sqrt{\frac{\pi\lambda m}{M(2m+M)c}}.$$

29. A solid hemisphere, mass M, is moveable on a smooth horizontal plane, on which its flat surface rests, about its vertical radius which is fixed. A very fine tube, the inside of which is smooth, is fixed on its surface, commencing at the vertex and cutting all the meridian lines at a constant angle a, and a particle mass m runs down this tube from the vertex; prove that just before the particle arrives at the horizontal plane the angular velocity w of the tube is such that

$$w^2(ma^2+Mk^2)(ma^2\cos^2 a+Mk^2)=2m^2ga^3\sin^2 a.$$

If the tube be closed at the end on the plane, and the particle be inelastic, what is the subsequent motion of the system?

30. A particle moves on a smooth parabola (of latus rectum $4a$) and is acted on at every point by a repulsive force from the focus which varies as the focal chord through the point; shew that the kinetic energy of the particle varies as

$$(r+4a)(r-2a)+2a^2\log\frac{r-a}{a}.$$

31. A particle P slides in a circular groove (radius a) ir a horizontal plane, and is attached to a point A of the groov by an elastic string, natural length a. Shew that the potentia energies of P at B in the two cases when the string i stretched directly from A to the particle, and when it lies ir the groove, are in the ratio $1 : (\pi - 1)^2$.

32. A thin uniform circular ring is set rotating wher hot about its centre, and its radius diminishes in cooling by a fraction proportional to the time. If no forces act, shew that the angle through which the ring has revolved in time is $\Omega at/r$, where a, Ω are initial radius and velocity, r the final radius.

33. On the surface formed by the revolution of a parabola about its directrix is placed a uniform elastic ring. I the plane of the ring be perpendicular to the directrix, shew that the time of a small oscillation is $2\sqrt{Ma\pi/T}$, where M is the mass of the ring, T the tension in the position of equilibrium, and $4a$ the latus rectum of the parabola.

34. Two particles m, m' slide in a smooth circular groove of radius a, whose plane is vertical, and are connected by a weightless rod whose length subtends an angle α at the centre of the circle; find an expression for the velocity o either particle at any instant, under given initial conditions and shew that the length of the simple equivalent pendulum for finite oscillations is

$$a\frac{m+m'}{m\cos\beta + m'\cos(\alpha-\beta)}, \text{ where } \tan\beta = \frac{m'\sin\alpha}{m+m'\cos\alpha}.$$

35. A fine string is placed in a smooth cycloidal tube o same length with open ends. The tube is placed in a vertical plane with base of cycloid horizontal and vertex upwards The string being slightly disturbed, investigate the motion completely, finding the velocity at any subsequent time Shew that when the string leaves the tube the velocity is that due to a fall from rest through eight-thirds of the diameter of the generating circle and that this takes place after a time $\sqrt{a/g}\cos^{-1}\sqrt{2/3}$ from commencement of motion.

36. A circular disc rolls down a rough curve in a vertical plane; if the initial and final positions of the centre of the disc be given, prove that, when the time of motion is the least possible, the curve is an involute of a cycloid.

37. AB is a vertical rod and a weight C of mass m is suspended from A by a weightless rod of length l, while a weight of mass M is connected to C by a second rod of length l, and can slide freely on AB. If the whole system be made to rotate about AB with uniform angular velocity ω, and θ be the angle which AC makes with the vertical when there is relative equilibrium, shew that

$$\cos\theta = \frac{g}{\omega^2 l}\left(1 + \frac{2M}{m}\right),$$

and that if the system be slightly disturbed the time of a small oscillation is

$$\frac{2\pi}{\omega}\left\{\frac{m^2(4M+m)\,\omega^4 l^2 - 4M(2M+m)^2 g^2}{m^3\omega^4 l^2 - m\,(2M+m)^2 g^2}\right\}^{\frac{1}{2}}.$$

38. Two centres of force of equal strength, one attractive and the other repulsive, are placed at two points, S and H, the law of force being that of the inverse square of the distance. Shew that a particle if placed anywhere on the plane bisecting SH at right angles will oscillate in a semi-ellipse of which S and H are the foci.

39. On a smooth wire bent into the form of an ellipse, a small ring is placed, which is attached to the foci by two elastic strings of the same material, whose natural lengths are equal to the shortest focal distance. Prove that, if the ring be placed at a very small distance from an extremity of the major axis, the velocity at any point varies as the distance from the major axis, and that the ratio of the pressures on the curve at the extremities of the axes is $b\,(1-e) : a$.

40. A particle is suspended so as to oscillate in a cycloid whose vertex is at the lowest point; if it begin to move from a point distant a from the lowest point measured along the curve, and the medium in which it moves gives a small resistance kv^2 to the acceleration, prove that before it next

comes to rest energy has been dissipated which is $8ka/3$ of its original value.

41. A tube in the form of a plane curve

$$(x^2+b^2)\,y+a^3=0$$

rotates freely about the axis of y which is vertical and measured upwards; the mass of the tube is M and its radius of gyration k; a heavy particle of mass P is capable of sliding in the tube, and the velocity of rotation is such that P is in equilibrium with respect to the tube: shew that its equilibrium will be stable or unstable according as Pb^2 is greater or less than Mk^2.

42. A heavy circular disc rests on a rough horizontal plane at A, a light elastic string is fastened at A, and, lying partly in a small groove in the rim of the disc, is just long enough to reach a point B vertically above A where also it is fastened. If the disc be drawn aside in its own plane and let go, find the velocity of its centre when passing through its initial position.

43. Three equal rods OA, OB, OC, of length a are jointed freely at O. The extremities A, B and C are connected by strings of length a and the system is placed upon a smooth table with A, B and C in contact with the table. The string AB is then cut; shew that, when the angle between OA and OB is 2θ,

$$\frac{96-144\sin^2\theta+64\sin^4\theta+13\tan^2\theta+9\tan^4\theta}{3-4\sin^2\theta}\left(\frac{d\theta}{dt}\right)^2$$
$$=\frac{144g}{a}\left(\sqrt{\frac{2}{3}}-\frac{\sqrt{4-\sin^2\theta}}{2}\right).$$

44. Four equal uniform rods, jointed freely so as to form a rhombus, are laid upon a smooth horizontal plane; the system is acted on by a repulsive force, tending from the centre of the rhombus, and varying inversely as the square of the distance: prove that, if $2a$ be the initial value of an acute angle of the rhombus, 2θ the angle between the rods containing this angle at the time t,

$$\left(\frac{d\theta}{dt}\right)^2 \text{ varies as } \log\tan\frac{\theta}{2}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cot\frac{a}{2}\cot\left(\frac{\pi}{4}-\frac{a}{2}\right).$$

45. A perfectly rough rod is gently placed with one end upon another rod of equal mass and in the same vertical plane, moving with the velocity $\sqrt{2gc}$ on a smooth table. If the initial inclination of the first rod to the horizon be α, and its length $2a$, shew that it will just rise to a vertical position if

$$2a(1-\sin\alpha)(5+3\cos^2\alpha)=3c\sin^2\alpha.$$

46. Six equal uniform rods are jointed together, so as to form a hexagon; every particle of each of two opposite rods repels every particle of the other rod with a force varying inversely as the square of the distance; prove that, if the rods be arranged in the form of a regular hexagon, and left to themselves, then if θ be the angle between either of the repelling rods, and one of the adjacent rods, after a time t,

$$(2+3\cos^2\theta)\,\dot\theta^2 \text{ varies as } (1+4\sin^2\theta)^{\frac{1}{2}}-2\sin\theta-2$$
$$+\sqrt{3}-\log\frac{(1+4\sin^2\theta)^{\frac{1}{2}}+1}{2\sqrt{3}\sin\theta}.$$

47. An endless string passes round the rim of an elliptic disc and a smooth peg in a horizontal plane in which the disc is revolving with uniform angular velocity ω about the peg, the major axis produced always passing through the peg. The distance between the foci is $2c$, that between the centre and the peg is a, and the radius of gyration round a vertical axis through the centre is k. Shew that when a slight disturbance is given, small oscillations take place, whose periods are $2\pi/n$, where

$$n^2k^2(a^2-c^2)=c^2\omega^2(a^2+k^2).$$

Tripos, June, 1882.

CHAPTER XIII.

EQUATIONS OF MOTION.

202. We have endeavoured in the preceding chapter to illustrate the application of the principles of the conservation of momenta, and of the conservation of energy, leading at once to differential equations of the first order for the determination of the motions of bodies.

We now proceed to consider the more general cases in which, by the action of external forces, momenta, or energy, or both, are imparted to, or taken from, a system, and also to explain the method of determining the actions and reactions between the bodies of a system.

It has been shewn in Chapter IV., as a result of Newton's Laws of Motion, that the aggregate of the time-fluxes of momenta of the bodies of a system is the exact equivalent of the aggregate of the acting forces.

The phrase 'effective forces' is usually applied to what we have called the time-fluxes of momenta, and, in this language, the system of effective forces is the exact equivalent of the system of acting forces.

If impulses be applied to a system, the aggregate of the changes of momenta of the particles of the system is the exact equivalent of the applied impulses.

203. Recapitulating from Chapter IV., we observe that, for the determination of the motion of a body or of a system, we have, as a result of the laws of motion, the following principles:

(1) The time-flux of the linear momentum of a system, in any assigned direction, is equal to the sum of the forces acting in that direction.

(2) The time-flux of the angular momentum of a system, about any assigned axis, is equal to the sum of the moments, about that axis, of the acting forces.

(3) The change, in any assigned direction, of the momentum of a system is equal to the sum of the applied impulses.

(4) The change of the angular momentum of a system, about any assigned axis, is equal to the sum of the moments, about that axis, of the applied impulses.

204. The linear momenta, parallel to the coordinate axes, of a system are

$$\Sigma(m\dot{x}),\quad \Sigma(m\dot{y}),\quad \Sigma(m\dot{z}),$$

and the angular momenta, about these axes, are

$$\Sigma m(y\dot{z}-z\dot{y}),\quad \Sigma m(z\dot{x}-x\dot{z}),\quad \Sigma m(x\dot{y}-y\dot{x}).$$

Hence it follows that the mathematical forms of the statements (1) and (2) are as follows,

$$(1)\quad \Sigma(m\ddot{x})=X,\quad \Sigma(m\ddot{y})=Y,\quad \Sigma(m\ddot{z})=Z,$$

$$(2)\quad \Sigma m(y\ddot{z}-z\ddot{y})=L,\ \Sigma m(z\ddot{x}-x\ddot{z})=M,\ \Sigma m(x\ddot{y}-y\ddot{x})=N.$$

Or these equations can be obtained, as in Art. 53, from the principle, Art. 46, that the system of the time-fluxes of momenta of the several particles of a system is the exact equivalent of the system of acting forces, and therefore that the sums of the components of the one system and their moments about the axes are the same as those of the other system.

In the case in which impulses are applied to a system, the mathematical forms of the statements (3) and (4) are

$$(3)\quad \Sigma m(u'-u)=P,\quad \Sigma m(v'-v)=Q,\quad \Sigma m(w'-w)=R,$$

$$\begin{aligned}(4)\quad &\Sigma m\{y(w'-w)-z(v'-v)\}=G,\\ &\Sigma m\{z(u'-u)-x(w'-w)\}=H,\\ &\Sigma m\{x(v'-v)-y(u'-u)\}=K,\end{aligned}$$

where P, Q, R, are the sums of the applied impulses in directions of the axes, and G, H, K the moments about the axes of those impulses.

205. *Independence of the motions of the centre of gravity and of the system relative to the centre of gravity.*

If ξ, η, ζ be the coordinates of G, the centre of gravity, the equations (1) become

$$\ddot{\xi}\Sigma(m)=X,\quad \ddot{\eta}\Sigma(m)=Y,\quad \ddot{\zeta}\Sigma(m)=Z,$$

shewing that the motion of G is the same as if the whole mass were concentrated into a particle at that point, with all the extraneous forces acting upon it.

Again, if we take x, y, z as the coordinates, relative to the centre of gravity, of a particle m of the system, its accelerations are $\ddot{x}+\ddot{\xi}$, $\ddot{y}+\ddot{\eta}$, $\ddot{z}+\ddot{\zeta}$, and, if L, M, N be the moments about the axes through G, of the acting forces, the equations (2) are replaced by

$$\Sigma m\{y(\ddot{z}+\ddot{\zeta})-z(\ddot{y}+\ddot{\eta})\}=L, \text{ \&c.},$$

or, since $\Sigma(my)=0$, and $\Sigma(mz)=0$,

$$\Sigma m(y\ddot{z}-z\ddot{y})=L, \text{ \&c.},$$

shewing that the motion relative to the centre of gravity is independent of the motion of the centre of gravity itself.

206. In the case of impulses being applied to the system, if the velocities of the centre of gravity change from a, b, c to a', b', c', the equations (3) become

$$M(a'-a)=P,\quad M(b'-b)=Q,\quad M(c'-c)=R,$$

M being the mass of the whole system.

Again, taking the instantaneous position of the centre of gravity as the origin, and taking u, v, w as the velocities of m relative to the centre of gravity, so that the actual velocities of m are $u+a$, $v+b$, $w+c$, the first of the equations (4) becomes

$$\Sigma m\{y(w'+c'-w-c)-z(v'+b'-v-b)\}=G,$$

or
$$\Sigma m\{y(w'-w)-z(v'-v)\}=G.$$

These results shew that the changes of motion of the centre of gravity, and the changes of angular momentum about axes through the centre of gravity, due to impulsive actions, are independent of each other.

207. To illustrate at once the use of these principles, consider the case of

A man walking on a large rough sphere, so as to make the sphere roll in a straight line on a horizontal plane, the man keeping himself at a constant angular distance (α) *from the highest point of the sphere.*

Take M, m as the masses of the sphere and the man, and let ω be the angular velocity of the sphere at any instant.

The velocity of the man is $a\omega$ parallel to the plane, and the time-flux of his momentum is therefore $ma\dot{\omega}$.

For the sphere the time-fluxes of the linear momentum, and of the angular momentum about the centre of gravity are

$$Ma\dot{\omega} \text{ and } M\tfrac{2}{5}a^2\dot{\omega}.$$

Hence taking moments about the horizontal axis, perpendicular to the motion, through the point of contact, we obtain

$$Ma^2\dot{\omega}+M\tfrac{2}{5}a^2\dot{\omega}+ma\dot{\omega}h=mga\sin\alpha,$$

or
$$\dot{\omega}(7Ma+5mh)=5mg\sin\alpha,$$

where h is the constant height, above the plane, of the centre of gravity of the man.

We may also solve this problem by finding the rate of change of the angular momentum, at any time, about the axis fixed in space which passes through the point of contact at that time.

To do this we observe that, at any time, the angular momentum of the system about the point of contact is equal to

$$\tfrac{7}{5}Ma^2\omega + mah\omega.$$

After the lapse of an interval δt of time, the angular velocity is $\omega + \delta\omega$, and the linear velocity of the centre of the sphere and of the man is $a(\omega + \delta\omega)$.

Observing that the distances from the assigned axis of the lines of acceleration of the centres are the same as at the time t, the angular momentum at the time $t + \delta t$ is

$$Ma^2(\omega + \delta\omega) + \tfrac{2}{5}Ma^2(\omega + \delta\omega) + ma(\omega + \delta\omega)h,$$

and the rate of change is therefore

$$\tfrac{7}{5}Ma^2\dot{\omega} + mah\dot{\omega},$$

which is equal to the moment $mga \sin \alpha$ of the acting forces.

If the reactions and frictions be required, we observe that the time-fluxes of the linear momenta of the sphere and the man are

$$Ma\dot{\omega} \text{ and } ma\dot{\omega},$$

so that, if F' be the friction between the sphere and plane, and R, F the reaction and friction between the man and the sphere,

$$Ma\dot{\omega} + ma\dot{\omega} = F'$$
$$ma\dot{\omega} = R \sin \alpha - F \cos \alpha$$
$$0 = R \cos \alpha + F \sin \alpha - mg,$$

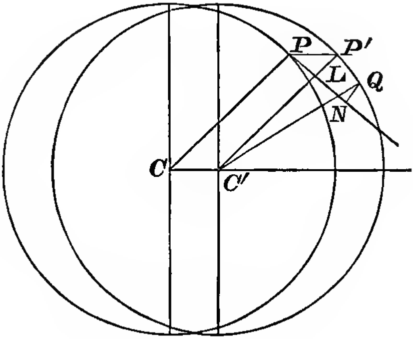
and these equations determine F, F' and R.

208. This system is not, from a purely mechanical point of view, a conservative system.

The work done by friction upon the sphere is not equal and opposite to the work done by friction upon the man, and the energy of motion at any time is due to the difference between the amounts of work done upon the two bodies.

Thus if the figure represent two consecutive positions, P being the point of contact, the horizontal distance

$PP' = a\delta\theta$, and if $P'C'Q = \delta\theta$, the point P of the sphere is carried to Q.



Hence, if $P'L$, QN be perpendiculars on the tangent at P, the work done by friction upon the sphere is $F \, . \, PN$, and the work done upon the man is $F \, . \, (-PL)$. The sum of these $= F \, . \, LN = Fa\delta\theta$, and the work done upon the system

$$= \int Fad\theta.$$

But, the acceleration of the man in the direction of the tangent at P being $a\dot{\omega}\cos\alpha$, it follows that

$$ma\dot{\omega}\cos\alpha = mg\sin\alpha - F,$$

and therefore the work done

$$= \int (mga\sin\alpha - ma^2\dot{\omega}\cos\alpha)\,d\theta.$$

This is transformed into energy of motion, or kinetic energy, the measure of which is

$$\tfrac{1}{2} M \, . \, \tfrac{7}{5} a^2\omega^2 + \tfrac{1}{2} ma^2\omega^2.$$

Equating these two expressions, we find that

$$\tfrac{1}{2} a^2\omega^2 \left(\tfrac{7}{5} M + m + m\cos\alpha\right) = mga\sin\alpha \, . \, \theta,$$

and, taking the time-fluxes of each side of this equation, we obtain the same equation for $\dot{\omega}$ as before.

The fact that kinetic energy is produced and increased is a proof that potential energy is being lost somewhere. The explanation is that the man's power of exerting himself is diminishing; he is getting tired, or, in other words, the man is a machine which has acquired potential energy by being wound up, and is running down.

209. *A heavy chain passes over a rough pulley, moveable about a horizontal axis through its centre, and the portions hanging down, which are of different lengths, are coiled up and held close to the ends of the horizontal diameter; if both coils be let go at the same instant what is the subsequent motion?*

The coils will fall freely, each leaving behind a straightened portion, and, at the time t, each will have fallen through the space $\frac{1}{2}gt^2$.

Take a for the radius of the pulley, b and c for the lengths of the portions coiled up, b being greater than c. Then if θ be the turn of the pulley in the time t, the lengths b', c' of the pieces in the coils at that time will be

$$b - (\tfrac{1}{2}gt^2 - a\theta) \text{ and } c - (\tfrac{1}{2}gt^2 + a\theta).$$

The velocity of each point of the straight pieces and of the portion in contact with the pulley will be $a\dot{\theta}$, and the velocity of each coil will be gt.

Let Mk^2 be the moment of inertia of the pulley, m the mass of the whole chain, and l its length, so that

$$l = b + c + \pi a.$$

Then the angular momentum of the system will be

$$Mk^2\dot{\theta} + \frac{m}{l}(gt^2 + \pi a)\, a^2\dot{\theta} + \frac{mb'}{l}gta - \frac{mc'}{l}gta,$$

or
$$Mk^2\dot{\theta} + \frac{m}{l}(gt^2 + \pi a)\, a^2\dot{\theta} + \frac{mgta}{l}(b - c + 2a\theta).$$

Taking the time-flux of this expression and equating it to the moment of the acting forces which is

$$\frac{mb'}{l}ga - \frac{mc'}{l}ga,$$

or
$$\frac{mga}{l}(b - c + 2a\theta),$$

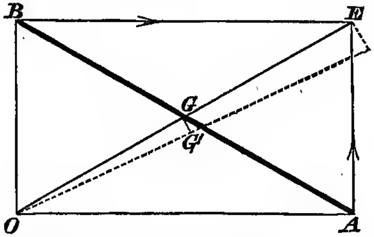
we obtain the equation

$$\left\{Mk^2 + \frac{ma^2}{l}(gt^2 + \pi a)\right\}.\ddot{\theta} + \frac{4mga^2t}{l}\dot{\theta} = 0,$$

from which, by integration and the introduction of initial conditions, $\dot{\theta}$ can be determined.

210. *Motion of a heavy rod sliding between a smooth vertical plane and a smooth horizontal plane in a vertical plane perpendicular to both.*

The angular velocity in any position is at once determined by the principle of energy, but the question is introduced for the sake of further illustrating the meaning of the phrase angular momentum.



If AB be the rod, and E the instantaneous centre of the motion, the velocity of G is $a\omega$, and the angular momentum about E is $ma^2\omega + mk^2\omega$, or $\frac{4}{3}ma^2\omega$.

After a time δt, the rod having turned through an angle $\delta\theta$, the perpendicular distance from E of the line of motion of G' is $2a\cos\delta\theta - a$, which to the first order of infinitesimals is equal to a.

Hence the new angular momentum about E is

$$ma^2(\omega + \delta\omega) + mk^2(\omega + \delta\omega),$$

and therefore the rate of change of the angular momentum about E is $\frac{4}{3}ma^2\dot{\omega}$ which is equal to $mga\sin\theta$;

$$\therefore\ \ddot{\theta} = \frac{3g}{4a}\sin\theta, \text{ and } \dot{\theta}^2 = \frac{3g}{2a}(\cos\alpha - \cos\theta).$$

This is one view of the case, and another is to observe that $a\omega^2$ and $a\dot{\omega}$ are the accelerations of G in the direction GO and perpendicular to it, so that the time-fluxes of momenta are $ma\omega^2$, $ma\dot{\omega}$, and $mk^2\dot{\omega}$, and the moments of these about E are equal to $mga \sin\theta$.

The reactions R, R' in the directions AE, BE are given by the equations,

$$R' = \frac{d}{dt}(ma\dot{\theta}\cos\theta) = ma\,(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta),$$

$$R - mg = \frac{d}{dt}(-ma\dot{\theta}\sin\theta) = -ma\,(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta).$$

It will be seen from these equations that R' vanishes and changes sign when

$$3\cos\theta = 2\cos\alpha,$$

shewing that the rod will then leave the vertical plane.

If it should be required to take moments about O, it will be seen that the angular momentum is $ma^2\omega - mk^2\omega$ and therefore that the equation of motion is

$$\tfrac{2}{3}ma^2\dot{\omega} = mga\sin\theta + R'2a\cos\theta - R2a\sin\theta,$$

but then the preceding equations must be employed for the elimination of R and R'.

211. *Motion of a rigid body about a fixed axis.*

If ω be the angular velocity of the body at any instant, its angular momentum is $M(k^2 + h^2)\,\omega$, h being the distance, OG, of the centre of gravity from the axis and Mk^2 the moment of inertia about the line through G parallel to the axis.

Hence if N be the moment of the acting forces,

$$M(k^2 + h^2)\,\dot{\omega} = N,$$

or, if θ be the inclination of the plane through G and the axis to a fixed plane through the axis,

$$M(k^2 + h^2)\,\ddot{\theta} = N.$$

If r be the distance of a particle m from the axis, its accelerations in the directions of r and perpendicular to it are respectively $-\omega^2 r$ and $r\dot{\omega}$.

Taking the fixed axis as the axis of z and OG as the axis of x, the components of these parallel to x and y are

$$-\omega^2 x - \dot{\omega} y \text{ and } -\omega^2 y + \dot{\omega} x.$$

Hence the time-fluxes of the momenta and their moments about the axes are respectively

$$\Sigma m\,(-\omega^2 x - \dot{\omega} y),\ \Sigma m\,(-\omega^2 y + \dot{\omega} x),\ o,$$

and $\Sigma m\,(\omega^2 yz - \dot{\omega} xz)$, $\Sigma m\,(-\omega^2 xz - \dot{\omega} yz)$, $\Sigma\{m\dot{\omega}\,(x^2 + y^2)\}$,

or, $$-M\omega^2 h,\ M\dot{\omega} h,\ o,$$

and $$D\omega^2 - E\dot{\omega},\ -E\omega^2 - D\dot{\omega},\ M\,(k^2 + h^2)\,\dot{\omega},$$

where D and E represent the quantities, Σmyz, and Σmzx.

The first three of these expressions are equal to the sums of the acting forces and of the reactionary stresses of the axis, and the next three are equal to the sums of the moments about the axes, of the same quantities*.

Suppose that the body is connected with the axis at two points at distances c and c' from O, and that U, V, W, and U', V', W' are the stresses upon the axis at these points.

Then, if X, Y, Z, L, M, N be the sums and moments of the acting forces, we shall have the equations

$$\begin{aligned}
-M\omega^2 h &= X - U - U', \\
M\dot{\omega} h &= Y - V - V', \\
o &= Z - W - W', \\
D\omega^2 - E\dot{\omega} &= L - Vc - V'c', \\
-E\omega^2 - D\dot{\omega} &= M + Uc + U'c'.
\end{aligned}$$

212. *Impulses applied to a body in motion about a fixed axis.*

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* It will be seen in the next chapter that these expressions can be also obtained by first writing the expressions for the angular momenta and then employing the general expressions given in that chapter for the rates of change of the angular momenta.

If $\omega' - \omega$ be the change of angular velocity due to the application of impulses, and K the moment of the impulses, the change of angular momentum is given by the equation

$$M(k^2 + h^2)(\omega' - \omega) = K.$$

Taking the axes as in the last article, and observing that $m(\omega' - \omega) r$ is the change of momentum of the particle m in the direction perpendicular to r, of which the components are $-m(\omega' - \omega) y$, and $m(\omega' - \omega) x$, we find that the sums and moments of the changes of momenta are respectively

$$o, \; M(\omega' - \omega) h, \; o,$$

and $$-E(\omega' - \omega), \; -D(\omega' - \omega), \; M(k^2 + h^2)(\omega' - \omega).$$

Equating these expressions to the sums and moments of the applied impulses, and of the reactionary stresses, we can calculate the latter quantities.

The right-hand members of the equations in the preceding article will be the expressions for the sums and moments, if the symbols employed be supposed to represent the applied impulses and the impulsive stresses on the axis at two points.

213. We have defined, in Art. 181, the centre of oscillation of a compound pendulum.

We can shew that the centres of oscillation and suspension E and O are convertible.

For $$OE \,.\, OG = OG^2 + k^2,$$

whence $$OG \,.\, EG = k^2,$$

shewing that O and E are convertible.

214. *Centre of percussion.*

If a single impulse can be applied to a rigid body which is capable of motion about a fixed axis, so as to produce no impulsive stress on the axis, the line of the impulse is called the line of percussion, and the point in which it meets the plane through G perpendicular to the axis is called the centre of percussion.

If ω be the angular velocity produced the sums and moments of the changes of momenta will be

$$o,\ M\omega h,\ o,\ -E\omega,\ -D\omega,\ M(k^2+h^2)\omega;$$

OG being the axis of x, and Oz the fixed axis as in Art. 211.

The single impulse P must therefore be in the direction of the axis of y, and, if ξ, ζ be the coordinates of the point in which its line of action meets the plane xy, we must have

$$-E\omega=-P\zeta,\quad -D\omega=o,\quad M(k^2+h^2)\,\omega=P\xi.$$

Hence it follows, as a necessary condition for the existence of a centre of percussion, that D, or $\Sigma\,(myz)$, must vanish, and that, if there is a centre of percussion, its distance from the axis is the same as that of the centre of oscillation.

When E, or $\Sigma\,(mzx)$, vanishes, the centres of oscillation and percussion are coincident.

215. *Motion of two heavy rods AB, BC jointed at B, and swinging in a vertical plane about the end A which is jointed to a fixed horizontal axis.*

If θ and ϕ be the inclinations of AB and BC to the vertical, the angular momentum of the system about A, $2a$ and $2b$ being the lengths of the rods, is

$$m\frac{4a^2}{3}\dot\theta+m'\frac{b^2}{3}\dot\phi+m'b\dot\phi\,\{b+2a\cos(\phi-\theta)\}$$
$$+\,m'2a\dot\theta\,\{2a+b\cos(\phi-\theta)\},$$

and the time-flux of this expression

$$=-mga\sin\theta-m'g\,(2a\sin\theta+b\sin\phi).$$

Next, the accelerations of G being compounded of $b\ddot\phi$ and $b\dot\phi^2$ perpendicular and parallel to GB, and of $2a\ddot\theta$ and $2a\dot\theta^2$ perpendicular and parallel to BA, we obtain by taking moments about B for the rod BC, and dividing by m'

$$b^2\ddot\phi+\frac{b^2}{3}\ddot\phi+2a\ddot\theta\,b\cos(\phi-\theta)$$
$$+\,2a\dot\theta^2 b\sin(\phi-\theta)=-gb\sin\phi,$$

or $\frac{4}{3}b\ddot\phi+2a\,\{\ddot\theta\cos(\phi-\theta)+\dot\theta^2\sin(\phi-\theta)\}=-g\sin\phi.$

Or for a second equation we might have expressed the constancy of the energy which is

$$\tfrac{2}{3}ma^2\dot{\theta}^2 + \tfrac{1}{6}m'b^2\dot{\phi}^2 + \frac{m'}{2}\{b^2\dot{\phi}^2 + 4a^2\dot{\theta}^2 + 4ab\dot{\phi}\dot{\theta}\cos(\phi - \theta)\}$$
$$- mga\cos\theta - m'g(2a\cos\theta + b\cos\phi).$$

In either case we obtain two equations for the determination of θ and ϕ.

The horizontal and vertical components of the momentum of the system are respectively

$$ma\dot{\theta}\cos\theta + m'b\dot{\phi}\cos\phi + m'2a\dot{\theta}\cos\theta,$$

and $$ma\dot{\theta}\sin\theta + m'b\dot{\phi}\sin\phi + m'2a\dot{\theta}\sin\theta.$$

The time-flux of the first of these is the horizontal component of the stress at A, and the time-flux of the second increased by the weight of the rods is the vertical component.

In the same manner the stress at B is determined by writing down the horizontal and vertical components of the momentum of the rod BC.

216. *Motion of a heavy sphere, the centre of gravity of which is eccentric, on a smooth horizontal plane.*

If the sphere have no initial kinetic energy, the centre of gravity G moves in a vertical line GO, and if $OG = y$, the equation of moments about C or about any point in the vertical line CP through the point of contact P is, with the notation of Art. 189,

$$m\ddot{y}c\sin\theta + mk^2\ddot{\theta} = -mgc\sin\theta,$$

where $$y = a - c\cos\theta.$$

Substituting, multiplying by $2\dot{\theta}$ and integrating we obtain

$$\dot{\theta}^2(k^2 + c^2\sin^2\theta) = C + 2gc\cos\theta,$$

which is the equation of energy.

The reaction, R, at P is given by the equation

$$R - mg = m\ddot{y}.$$

If the plane be rough so that the sphere rolls, the equation of moments about P is

$$mk^2\ddot{\theta} + m\ddot{x}y + m\ddot{y}c\sin\theta = -mg \,.\, c\sin\theta,$$

where $$x = a\theta - c\sin\theta, \text{ and } y = a - c\cos\theta,$$

making the substitutions and integrating we obtain the result of Art. 189.

The horizontal and vertical reactions at P are given by the equations $m\ddot{x} = -F$, $m\ddot{y} = R - mg$.

217. *Motion of a plane lamina, of any given shape, on a smooth plane, when a given point of the lamina is made to move in a given manner, and the lamina is besides acted upon by known forces.*

If A be the given point, and G the centre of inertia of the lamina, take θ as the inclination of AG to a fixed line in the plane, so that $\dot{\theta}$ is the angular velocity of the lamina.

If f and f' be the accelerations of the point A in the direction AG and perpendicular to it, which are known functions of the position of A, or of the time, the accelerations of G in the same directions are $f - a\dot{\theta}^2$, and $f' + a\ddot{\theta}$, and the time-flux of the angular momentum about G is $Mk^2\ddot{\theta}$.

Hence, if N be the moment about A of the acting forces,

$$M(f' + a\ddot{\theta})\,a + Mk^2\ddot{\theta} = N,$$

the equation which determines the angular motion of AG.

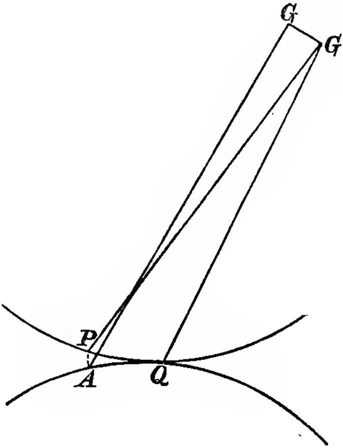
If P, Q be the requisite constraining stresses at A, and X, Y the resultant forces, in the direction AG and perpendicular to it,

$$M(f - a\dot{\theta}^2) = X + P, \text{ and } M(f' + a\ddot{\theta}) = Y + Q.$$

218. *Motion of a plane lamina, bounded by a curve, rolling on a fixed curve under the action of given forces.*

We have first to solve the kinematical question of the time-flux of the angular momentum about the point of contact.

The angular momentum being the sum of the angular momenta due to the rotation and the motion of the centre of gravity, the first part is $Mk^2\omega$.



For the second, taking ω as the angular velocity when the point P of the rolling curve is in contact with the point A of the fixed curve, and Q as the consecutive point of contact, the angular momentum at $A = Mr^2\omega$, if

$$AG = PG' = r.$$

In the consecutive position, when the motion of G' is perpendicular to QG', the angular momentum about A

$$= M(r + \delta r)(\omega + \delta\omega)\, r,$$

remembering that AP is an infinitesimal of the second order, and the difference

$$= Mr\,(r\delta\omega + \omega\delta r).$$

Hence the time-flux of the angular momentum due to the motion of G

$$= Mr^2\dot{\omega} + M\omega r\dot{r},$$

and consequently the equation of motion is

$$M(k^2+r^2)\dot{\omega}+M\omega r\dot{r}=L,$$

L being the moment about P of the acting forces.

If ρ, ρ' be the radii of curvature at P of the rolling curve and the fixed curve, and if the arc $AQ=\delta s$,

$$\omega\delta t=\frac{\delta s}{\rho}+\frac{\delta s}{\rho'},$$

and, if θ be the angle between the line AG and the normal at A,

$$\omega r\dot{r}=-\omega r\sin\theta\dot{s}=-\omega^2 r\sin\theta\frac{\rho\rho'}{\rho+\rho'};$$

so that the equation of motion takes the form

$$M(k^2+r^2)\dot{\omega}-M\omega^2 r\sin\theta\frac{\rho\rho'}{\rho+\rho'}=L.$$

If the fixed curve be a straight line

$$\omega\delta t=\frac{\delta s}{\rho},$$

and the equation is then

$$M(k^2+r^2)\dot{\omega}-M\omega^2\rho r\sin\theta=L.$$

219. The result of the preceding article may be otherwise obtained.

In Art. 25, it is shewn that the acceleration of the point P, when at A, in direction of the normal at A is

$$\omega^2\rho\rho'/(\rho+\rho').$$

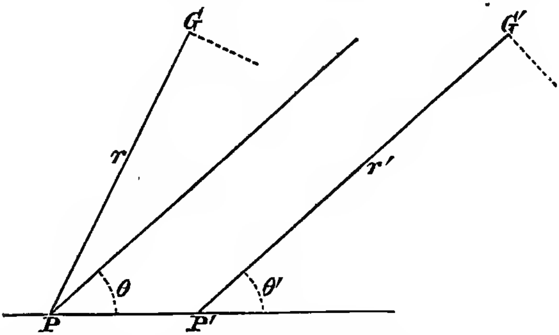
The acceleration of G relative to A, in the direction perpendicular to AG is $r\dot{\omega}$, and therefore the actual acceleration of G perpendicular to AG is

$$r\dot{\omega}-\omega^2\rho\rho'\sin\theta/(\rho+\rho').$$

Hence, equating momenta about A,

$$Mk^2\dot{\omega}+Mr\{r\dot{\omega}-\omega^2\rho\rho'\sin\theta/(\rho+\rho')\}=L.$$

220. The investigation of the two preceding articles is the infinitesimal case of the following general statement.



If Q be the linear momentum of a body, in motion in one plane, when P is the instantaneous centre, and Q' at a subsequent time when P' is the instantaneous centre, G being the centre of inertia, the change of the angular momentum about P is

$$Mk^2(\omega' - \omega) + Q'(r' + PP' \cos\theta) - Qr.$$

221. *As a particular case consider the motion of a heavy uniform circular disc of radius c rolling on the curve,* $s = cf(\phi)$, *starting from the highest point, from which s and* ϕ *are measured.*

In this case, $\theta = 0$,

and
$$\omega = \left\{\frac{1}{c} + \frac{1}{cf'(\phi)}\right\} cf'(\phi)\,\phi = (1 + f'\phi)\,\dot{\phi},$$

and the equation of motion becomes

$$M\frac{3c^2}{2}\{[1 + f'(\phi)]\,\ddot{\phi} + f''(\phi)\,\dot{\phi}^2\} = Mgc \sin\phi.$$

Suppose the curve to be a cycloid, $s = c \sin\phi$;

then
$$\cos\frac{\phi}{2}\,\ddot{\phi} - \sin\frac{\phi}{2}\,\dot{\phi}^2 = \tfrac{2}{3}\,.\,\frac{g}{c}\sin\frac{\phi}{2},$$

the integral of which is

$$3c\dot{\phi}^2 \cos^4\frac{\phi}{2} = 2g\left(1 - \cos^4\frac{\phi}{2}\right),$$

and therefore $$\omega^2 = \frac{8g}{3c}\left(1 - \cos^4\frac{\phi}{2}\right).$$

To find the pressure we have the equation,

$$M\frac{c^2\omega^2}{c+\rho} = Mg\cos\phi - R,$$

shewing that R vanishes, and therefore that the disc flies off, when $5\cos\phi = 3$.

At the instant of flying off, $\omega^2 = 24g/25c$, and the velocity of the centre of the disc is $2\sqrt{6gc}/5$.

222. *Change of motion produced in a lamina, moving in any manner in its plane, produced by its impact on a rough curve.*

We suppose the roughness to be so great that there is no sliding, and we have simply to express the fact that the angular momentum round the point of contact is unchanged.

P being the point, let v be the component perpendicular to GP of the velocity of G, ω and ω' the angular velocities just before and after the impact; then

$$M(k^2 + r^2)\,\omega' = Mk^2\omega + Mvr.$$

Suppose for example that the disc of the preceding article, just after flying off the cycloidal arc impinges on a fixed rough peg just beneath its lowest point.

In this case

$$\frac{3c^2}{2}\,\omega' = \frac{c^2}{2}\,\omega + c^2\omega\cos\phi,$$

from which we obtain $15\omega' = 11\omega$.

The disc will then turn round the peg as a fixed point, and the equations of motion will determine the angle through which it turns before leaving the peg.

223. In general if a rigid body, or system of any kind, be in motion, and if a straight line of the system suddenly become fixed, the angular momentum of the system about the axis is unchanged.

In the case of a single rigid body this at once determines the angular velocity about the axis.

Thus, if Q be the linear momentum of a rigid body perpendicular to the axis which becomes fixed, p the distance from the axis of its centre of inertia G, and H the angular momentum about the line through G parallel to the axis, the angular velocity is given by the equation

$$Mk^2\omega = Qp + H,$$

Mk^2 being the moment of inertia about the axis which becomes fixed.

224. *Tendency of a rod in motion to break at any assigned point.*

Imagine a rod of small section and of any shape, its axis however being a plane curve, to be in motion in that plane under the action of forces in the plane.

Taking any cross section, through any point P of the axis, the stress at this section, that is, the action and reaction between the two parts of the rod separated by the cross section, may be represented by two forces T and N at P in directions of the tangent and normal to the axis, and a couple G in the plane of the axis.

The velocities and accelerations of the various points of the axis having been previously determined, the quantities T, N, and G can be found by writing down the equations of motion of either of the two parts of the rod, including T, N, and G amongst the acting forces.

Now a rod may break in three ways; the internal molecular forces may not be sufficient to withstand the force T in direction of the tangent, or they may give way in direction of the normal, or the moment about P of the molecular forces may be overpowered by the couple G.

In other words the rod may break by tearing, by shearing, or by snapping, and the quantities T, N, and G are, respectively, the measures of the tendencies to break in these three ways.

To illustrate, take the case of a heavy straight rod swinging in a vertical plane about one end, and examine the tendency to break at the middle point.

Writing down the equations of motion of the lower half of the rod, and taking r as the distance from the axis of a point in the rod, we obtain

$$\int_a^{2a} m\frac{dr}{2a} r\ddot{\theta} = N - \tfrac{1}{2}mg\sin\theta,\quad \int_a^{2a} m\frac{dr}{2a} r\dot{\theta}^2 = T - \tfrac{1}{2}mg\cos\theta,$$

$$\int_a^{2a} m\frac{dr}{2a} r(r-a)\,\ddot{\theta} = G - \tfrac{1}{4}mga\sin\theta,$$

where $\dot{\theta}$ and $\ddot{\theta}$ are known functions of θ, and these equations determine the values of T, N, and G at the middle point of the rod.

225. If a rod have its state of motion suddenly changed by impulsive action, impulsive stresses are created at all points of the rod, and the method of determining them is the same as in the previous case.

If for instance the free end of the swinging rod, supposed inelastic, be suddenly stopped by impinging against a fixed surface, and if T, N, and G then represent impulses, the equations are

$$T = 0,\ \int_a^{2a} -m\frac{dr}{2a} r\dot{\theta} = N - P,$$

$$\int_a^{2a} -m\frac{dr}{2a} r(r-a)\,\dot{\theta} = G - \tfrac{1}{2}Pa,$$

where P, the impulse at the free end, is given by the equation,

$$2ma\dot{\theta} = 3P.$$

226. *Determination of initial stresses, and initial accelerations, when some of the constraints of a system, previously in equilibrium, are removed.*

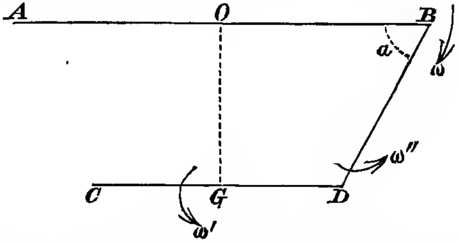
In such cases the equations of motion should be written down for the configuration of the system at the instant of release from constraint.

These equations, in combination with the kinematical relations of the system, will be sufficient for the determination of the required stresses.

The equations are simplified by the fact that the linear and angular velocities are zero, so that radial and transversal accelerations take the forms $\ddot{r}$ and $r\ddot{\theta}$, and normal accelerations are evanescent*.

EXAMPLES. (1) *Two rods AB, CD, of lengths 2a and 2b, are connected by equal strings AC, BD, of length c, and the system is supported, with the rods horizontal, by a fixed horizontal axis through the middle point of AB; if one string AC be cut, it is required to find the initial tension of the other.*

Let ω, ω', and ω'' be the initial angular accelerations, in the directions figured, of the two rods and the string.



The vertical acceleration of G, the centre of gravity of CD,

$$= b\omega' + c\omega'' \cos\alpha + a\omega,$$

and its horizontal acceleration $= c\omega'' \sin\alpha$.

Hence the equations of motion are

$$m\frac{a^2}{3}\omega = Ta\sin\alpha,$$

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* Some illustrations of the method of finding initial stresses are given in an article in the *Mathematical Messenger* for 1866.

$$m'(b\omega' + c\omega'' \cos \alpha + a\omega) = m'g - T \sin \alpha,$$

$$m'c\omega'' \sin \alpha = T \cos \alpha,$$

$$m' \frac{b^2}{3} \omega' = T . b \sin \alpha,$$

which determine the tension and the angular accelerations.

(2) *A heavy rod, of length 2a, is supported against a smooth fixed sphere by a horizontal string fastened to its upper end A, and also to the highest point of the sphere; if the string be cut it is required to find the initial pressure on the sphere.*

If α be the angular distance of the vertex from P the point of contact, it will be found that $PG = a \sin^2\alpha$. Observing that the acceleration of P is wholly tangential, and taking ω as the initial angular acceleration of the rod, it follows that ωPG is the acceleration of G perpendicular to the rod, and therefore taking moments about P

$$m \frac{a^2}{3} \omega + mPG^2\omega = mgPG \cos \alpha.$$

We have also $$m\omega PG = mg \cos \alpha - R,$$

and we obtain $$R(1 + 3 \sin^4 \alpha) = mg \cos \alpha.$$

227. *Determination of the initial radii of curvature of the paths of assigned points of a system, when the system is set in motion in any given manner, or, being in a state of equilibrium, has some of its constraints removed.*

If the velocity and direction of motion of an assigned point of the system be known, the expression for the normal acceleration, v^2/ρ, determines the curvature, $1/\rho$, for the acceleration of the point in the direction of the normal to its path is obtainable from the equations of motion of the system.

EXAMPLES. (1) *Two particles, m and μ, are connected by a string passing over a smooth fixed horizontal rail, and, the portions of string, of lengths a and b, being vertical, the particles are projected horizontally, in opposite directions perpendicular to the rail.*

The initial equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = mg - T, \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0,$$

$$\mu(\ddot{\rho} - \rho\dot{\phi}^2) = \mu g - T, \quad \mu(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) = 0.$$

If u and v be the velocities of projection, then, initially,

$$r = a,\ \rho = b,\ \dot{r} = 0,\ \dot{\rho} = 0,\ a\dot{\theta} = u,\ b\dot{\phi} = v,$$

and therefore $\ddot{\theta} = 0,\ \ddot{\phi} = 0,$

$$m\left(\ddot{r} - \frac{u^2}{a}\right) = mg - T, \quad \mu\left(\ddot{\rho} - \frac{v^2}{b}\right) = \mu g - T,$$

and these equations, with the equation $\ddot{r} + \ddot{\rho} = 0$, determine $\ddot{r}$, $\ddot{\rho}$, and T.

The initial radii of curvature R, R' of the paths of m and μ are given by the equations

$$m\frac{u^2}{R} = T - mg, \quad m\frac{v^2}{R'} = T - \mu g,$$

assuming the concavities to be upwards.

If T is less than either mg, or μg, the concavity in that case will be downwards.

(2) *A rod AB is moveable in a vertical plane about the end A, and a string BC carries a heavy particle at C; the particle is held in a given position in the vertical plane through the rod, and is projected in the direction perpendicular to BC in the vertical plane.*

If θ and ϕ be the inclinations of AB and BC to the vertical at the moment of projection, the initial equations of motion are

$$M\frac{4a^2}{3}\ddot{\theta} + mb\ddot{\phi}\{b + 2a\cos(\phi - \theta)\} - mb\dot{\phi}^2 2a\sin(\phi - \theta)$$

$$+ m2a\ddot{\theta}\{2a + b\cos(\phi - \theta)\} + m2a\dot{\theta}^2 b\sin(\phi - \theta)$$

$$= Mga\sin\theta + mg(2a\sin\theta + b\sin\phi),$$

and $\quad b\ddot{\phi} + 2a\ddot{\theta}\cos(\phi - \theta) + 2a\dot{\theta}^2\sin(\phi - \theta) = -g\sin\phi,$

where $2a$ and b are the lengths of AB and BC.

Initially, if u be the velocity of projection,

$$\dot{\theta}=0, \text{ and } u=b\dot{\phi},$$

and the preceding equations determine $\ddot{\theta}$ and $\ddot{\phi}$.

The acceleration of the particle in the direction CB is initially $b\dot{\phi}^2-2a\ddot{\theta}\sin(\phi-\theta)$, and this is equal to u^2/ρ if ρ be the initial radius of curvature of the path of C.

228. If a system have initially no motion, and we wish to find the initial curvature of the path of any assigned point of the system, we must first find the initial direction of motion, and then, observing the small displacements which take place in a very short time, we can sometimes obtain the curvature by an immediate application of the Newtonian expression for the diameter of curvature, viz. $(\text{arc})^2 \div$ perpendicular subtense. Sometimes however it is necessary to take the analytical expression, in Cartesian or polar co-ordinates, or in some other system, and to expand, in ascending powers of the time, the various terms contained in these expressions. An illustration of each case will be sufficient to explain the methods.

(1) *Two rods, AB, BC, freely jointed together at B, and moveable about the end A, are held in a horizontal position so as to form a straight line ABC, and are then let go; it is required to find the initial curvature of the path of C.*

Let μ, m be the masses and $2a$, $2b$ the lengths of AB and BC, then, if ω and ω' be the initial angular accelerations of AB and BC, it can be shewn, by taking moments about A for the system and about B for the rod BC, and combining the equations, that

$$\frac{\omega}{\omega'}=-\frac{b}{a}\,\frac{m+2\mu}{\mu}.$$

Now supposing that, after a short time t, θ and ϕ are the inclinations of AB and BC to the horizontal, it follows that the horizontal and vertical displacements of C are

$$2a+2b-2a\cos\theta-2b\cos\phi, \text{ and } 2a\sin\theta+2b\sin\phi,$$

or, approximately,

$$a\theta^2+b\phi^2 \text{ and } 2a\theta+2b\phi.$$

The initial tangent to the path of C is vertical, and consequently

$$2\rho = \text{Limit of } \frac{4\,(a\theta + b\phi)^2}{a\theta^2 + b\phi^2}.$$

The first approximations to θ and ϕ are

$$\theta = \tfrac{1}{2}\omega t^2,\ \phi = \tfrac{1}{2}\omega' t^2,$$

and we hence obtain

$$\frac{\rho}{2ab} = \frac{(m+\mu)^2}{b\,(m+2\mu)^2 + a\mu^2}.$$

(2) *A plane lamina, which is moveable about a horizontal axis on its plane, is inclined to the horizontal at an angle α, and a heavy particle m is placed upon it at a distance c from the axis and below the axis; it is required to determine the initial curvature of the path of the particle.*

If θ be the inclination at any time, the equations of motion are

$$Mk^2\ddot{\theta} + mr\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mgr\cos\theta,$$
$$\ddot{r} - r\dot{\theta}^2 = g\sin\theta.$$

The expression for the radius of curvature is

$$\frac{(r^2\dot{\theta}^2 + \dot{r}^2)^{\frac{3}{2}}}{r^2\dot{\theta}^3 + 2\dot{r}^2\dot{\theta} - r\,(\dot{\theta}\ddot{r} - \dot{r}\ddot{\theta})},$$

and we have to expand in powers of t, the various terms of this expression, which can be effected by Maclaurin's Theorem.

Taking ml^2 to represent $Mk^2 + mc^2$, we obtain, from the equations of motion,

$$\ddot{r}_0 = g\sin\alpha,\ \ddot{\theta}_0 = \frac{gc\cos\alpha}{l^2},\ \ddot{r}_0 = 0,\ \dddot{\theta}_0 = 0,$$

$$r_0^{\text{iv}} = \frac{g^2c\cos^2\alpha}{l^2}\left(1 + \frac{2c^2}{l^2}\right),\ \theta_0^{\text{iv}} = \frac{g^2\sin\alpha\cos\alpha}{l^2}\left(1 - \frac{7c^2}{l^2}\right).$$

Now $$\dot{\theta} = \ddot{\theta}_0 t + \ldots, \ \dot{r} = \ddot{r}_0 t + \ldots,$$

and $$\dot{\theta}\ddot{r} - \dot{r}\ddot{\theta} = (\ddot{\theta}_0 r_0^{\text{iv}} - \ddot{r}_0 \theta_0^{\text{iv}}) \frac{t^3}{3} + \ldots\ldots\ldots,$$

and therefore

$$\rho_0 = \frac{(\ddot{r}_0^2 + c^2 \ddot{\theta}_0^2)^{\frac{3}{2}}}{c^2 \ddot{\theta}_0^3 + 2\ddot{r}_0^2 \ddot{\theta}_0 - \frac{c}{3} (\ddot{\theta}_0 r_0^{\text{iv}} - \ddot{r}_0 \theta_0^{\text{iv}})}$$

and, making the requisite substitutions, we obtain ρ_0 in terms of α, l, and c.

If $\alpha = 0$, it will be found that

$$\rho_0 = \frac{3c^3}{c^2 - l^2} = -\frac{3mc^3}{Mk^2},$$

the negative sign shewing that the concavity of the initial path is outwards from the axis.

229. *Application of the principle of virtual work.*

If at any instant the geometrical configuration of a system be contemplated, and if a geometrical displacement be imagined, the virtual work of the time-fluxes of momenta, or of the effective forces, and of the acting forces will be the same.

We must however include in the phrase acting forces any internal forces such as the tensions of elastic strings, or sliding frictions, by means of which work is done on the system.

Considering a single rigid body, i.e. a material system of invariable form, if F be the time-flux of the linear momentum in any direction, and δs the displacement of the centre of gravity in that direction; and if K be the time-flux of the angular momentum about an assigned axis through the centre of gravity, and $\delta\phi$ the angular displacement about the axis, then the corresponding portions of the virtual work are $F\delta s$ and $K\delta\phi$.

(1) Consider for example the case of the two rods in Art. 215.

Taking x and y as the horizontal and vertical co-ordinates of G, we obtain

$$m\frac{4a^2}{3}\ddot{\theta}\delta\theta + m'\frac{b^2}{3}\ddot{\phi}\delta\phi + m'\ddot{x}\delta x + m'\ddot{y}\delta y = m'g\delta y - mga\sin\theta\delta\theta.$$

Now $x = 2a\sin\theta + b\sin\phi$, and $y = 2a\cos\theta + b\cos\phi$, from which δx and δy are obtained in terms of $\delta\theta$ and $\delta\phi$, and observing that $\delta\theta$ and $\delta\phi$ are arbitrary quantities and independent of each other, their coefficients must each vanish, and we thus obtain the equations for the determination of θ and ϕ.

EXAMPLE. (2) *Motion of an extensible circular ring, placed horizontally over a smooth surface of revolution the axis of which is vertical.*

Let s represent the distance along a meridian arc from a fixed level to the ring, r its radius and z the depth of its plane. The accelerations of any point of the ring down the meridian arc and perpendicular to it are $\ddot{s}$ and $\dot{s}^2/\rho$, and therefore, if we imagine a displacement by slightly shifting the ring downwards on the surface the equation of virtual work is, m being the mass of the ring,

$$m\ddot{s}\delta s = mg\delta z - T\delta(2\pi r),$$

or

$$m\ddot{s} = mg\frac{dz}{ds} - 2\pi T\frac{dr}{ds},$$

an equation which can also be obtained by considering the meridional motion of an element of the ring.

Observing that $T = \lambda(r - a)/a$, and that $r = f(z)$, this equation determines the acceleration along any meridian.

(3) *Four equal rods, of length 2a, are jointed together in the form of a square OADB, and suspended from the joint O, the square form being maintained by a string OD; if the string be cut it is required to find the change of stress at O.*

Take ω as the expression for the initial angular acceleration of each rod; then if G and H be the centres of gravity of OA and AD, the initial vertical accelerations of G and H are

$$a\omega/\sqrt{2} \text{ and } 3a\omega/\sqrt{2};$$

and the initial horizontal accelerations are each $a\omega/\sqrt{2}$, also the time-flux of the angular momentum of each rod is

$$ma^2\omega/3.$$

If we imagine D pulled through a small space, so as to displace each rod angularly through the small angle θ, the linear displacements of G and H are, vertically, $a\theta/\sqrt{2}$, $3a\theta/\sqrt{2}$, and horizontally each is $a\theta/\sqrt{2}$; also the vertical displacement of the centre of gravity K of the system is $a\theta\sqrt{2}$.

Hence we obtain

$$2m\,\frac{a^2\omega\theta}{2} + 2m\,\frac{9a^2\omega\theta}{2} + 4m\,\frac{a^2\omega\theta}{2} + 4m\,\frac{a^2\omega\theta}{3} = 4mga\theta\sqrt{2},$$

from which $10a\omega = 3g/\sqrt{2}$, and therefore the acceleration of K is $3g/5$, shewing that the pressure on the point of support is instantaneously diminished by three-fifths of the weight of the system.

If P be the horizontal stress at D, it can be determined by giving the rod AD a small arbitrary twist, θ, about A, breaking the connection at D.

The equation of virtual work will be

$$m\frac{a^2}{3}\omega\delta\theta + \frac{3ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}} - \frac{ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}} = mg\,\frac{a\theta}{\sqrt{2}} + Pa\theta\sqrt{2},$$

from which $P = -\dfrac{mg}{10}$.

If Q and R be the actions at A upon OA in the directions DA and OA, these quantities may be found by giving DA a twist about D, and OA a twist about O, breaking in each case the connection at A.

The equations obtained are

$$\frac{ma^2\omega}{3}\theta - \frac{3ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}} + \frac{ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}} = R2a\theta - mg\,\frac{a\theta}{\sqrt{2}},$$

$$-m\frac{a^2\omega}{3}\theta-\frac{ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}}-\frac{ma\omega}{\sqrt{2}}\cdot\frac{a\theta}{\sqrt{2}}=Q2a\theta-mg\frac{a\theta}{\sqrt{2}},$$

and from these we find that

$$R=\tfrac{3}{20}mg\sqrt{2}\text{ and }Q=\tfrac{1}{20}mg\sqrt{2}.$$

In solving this question our object has been to illustrate the use of the principle of work, but the same result may be obtained from the initial equations of motion of the rods, which are

$$\frac{4ma^2\omega}{3}=\frac{mga}{\sqrt{2}}-2aQ,$$

$$\frac{3ma\omega}{\sqrt{2}}=mg+\frac{Q-R}{\sqrt{2}},\quad\frac{ma\omega}{\sqrt{2}}=\frac{Q+R}{\sqrt{2}}-P,$$

$$\frac{ma^2\omega}{3}=Ra+\frac{Pa}{\sqrt{2}}.$$

EXAMPLES.

1. A smooth sphere is at rest on a smooth horizontal plane, and an equal sphere is placed gently upon it, so as to be in contact very nearly at the highest point; prove that the centre of the upper sphere will describe a portion of the arc of an ellipse, and that when θ is the inclination to the vertical of the line of centres,

$$a\dot{\theta}^2(1+\sin^2\theta)=2g(1-\cos\theta).$$

Shew that the spheres will part company when

$$\cos\theta=\sqrt{3}-1.$$

2. Two equal smooth spheres rest in contact on a smooth horizontal plane and another sphere is placed between them, so that motion ensues. Find the pressure between them at any time in terms of the inclination to the horizon of the lines joining their centres.

3. If two weights be suspended by a weightless string, which passes over a rough circular cylinder of given mass, the space described by either in any time is independent of the radius of the cylinder.

4. Two equal smooth spheres are placed one upon the other and both in contact with a smooth vertical wall. If the lower one just leave the wall, prove that they will separate when the line joining their centres is inclined to the horizon at an angle $\cos^{-1}(\frac{2}{3})$, the motion being supposed to take place in a vertical plane.

5. Four equal rods are jointed together so as to form a square *ABCD*, and the system is suspended from the point *A*, the square form being maintained by a string connecting *A* and *C*. Find the tension of the string.

If the string be cut, prove that during the subsequent motion,

$$a\dot{\theta}^2(1 + 3\sin^2\theta) = 3g\left(\cos\theta - \frac{1}{\sqrt{2}}\right),$$

$2a$ being the length of each rod, and θ the inclination to the vertical.

6. Three equal smooth balls are kept in contact with each other on a smooth horizontal plane by a string passing round them, and a fourth equal ball rests upon the three; if the string be cut, what is the initial change of pressure between the upper ball and each of the lower ones?

7. Four equal smooth inelastic circular discs, of radius a, are placed in one plane with their centres at the four corners of a square of which each side is $2a$. They attract one another with a force varying as the distance. A blow being given to one of them in the line of one of the diagonals of the square, investigate the whole of the subsequent motion.

8. A solid sphere resting on a smooth horizontal plane, is suddenly divided into two equal parts by a vertical plane through its centre. It is required to determine the initial horizontal pressure between the two parts and the initial vertical re-action of the plane.

9. The middle point of a uniform rod is fixed midway between two centres of force, which attract with a force varying inversely as the square of the distance. Prove that the time of a small oscillation is

$$\pi (a^2 - c^2) \sqrt{M}/\sqrt{3\mu ac},$$

where M is the mass of the rod, $2c$ its length, $2a$ the distance between the centres of force, and $\mu\delta x/r^2$ the attraction on an element δx of the rod at a distance r.

10. A rod of given length is formed into the quadrantal arc of a circle, and is made to rotate about an axis through one end perpendicular to its plane.

Supposing the arc to become suddenly fixed to its axis, find the measure of the tendency to break off; and shew that, if the rod were formed into a semi-circular arc, the tendencies to break off in the two cases would be compared by the ratio $4\pi - 8 : \pi$.

11. A rough cylinder rests on a horizontal plane. Find the least velocity of a second cylinder of given larger radius, which will, after impinging upon it, pass over it.

12. If a bullet of mass m be fired with velocity u perpendicular to the face of a block of wood of mass M, placed on a smooth horizontal plane, and remains just imbedded, prove that the angular velocity acquired by the block is

$$\frac{mbu}{(M+m)k^2 + m(a^2+b^2)},$$

where a is the distance of the centre of gravity of the block from the face struck, and b is the distance of the point struck from the foot of the perpendicular drawn from the centre of gravity of the block to the face.

13. A uniform rod of length $2a$ is rotating, in a vertical plane, about its middle point, which is fixed, with an angular velocity $\sqrt{6\pi g/a}$. At the instant the rod is horizontal, the ascending end is struck by a ball of equal mass, which was dropped from a height $3\pi a$; and when it is next horizontal,

the same extremity is struck by a second equal ball similarly dropped. The elasticity being perfect, determine the subsequent motion of the rod and balls.

14. A wire in the form of the portion of the curve $r = a(1 + \cos\theta)$ cut off by the initial line rotates about the origin with angular velocity ω; shew that the tendency to break at the point $2\theta = \pi$ is measured by $12\sqrt{2}m\omega^2a^3/5$, where m is the mass of a unit of length of the wire.

15. A rod rests horizontally upon two supports; if one support be suddenly withdrawn, find an equation to determine where the initial strain on the rod is greatest.

16. A rough cylinder of radius a loaded so that its centre of gravity is at a distance h from its axis is placed on a board of n times its mass, which can move on a smooth horizontal plane. Find the time of an oscillation when the system is slightly disturbed from its position of stable equilibrium, and prove that if l be the length of the simple equivalent pendulum

$$(n+1)(lh - k^2) = n(a-h)^2.$$

17. A circular disc is suspended from a fixed point by four equal strings symmetrically attached to its rim, and inclined at an angle α to the vertical. If two of the strings be cut, shew that the initial tension of each of the others is to the weight of the disc in the ratio

$$\cos\alpha : 1 + 5\cos^2\alpha.$$

18. A circular ring, mass M and radius a, lies on a smooth horizontal plane, and a fly, mass m, alighting upon it starts off and crawls round the ring, with a velocity v, which is uniform relative to the ring. Prove that the angular velocity of the ring $= \dfrac{mv}{(M+2m)a}$.

19. If the ring in the previous question be vertical and moveable about its centre of gravity, and if the fly start off and as before move uniformly relative to the ring, find its

angular velocity in any position, supposing the fly to start from the lowest point. Also find the least ratio of the masses in order that the fly may ever be at the highest point of the ring.

20. A flat square board *ABCD*, moveable about its centre on a smooth horizontal plane, has attached to *A* a fine string *AP* (=*AB*) which carries a particle *P*; *P* is projected horizontally from *B* at right angles to *AB*. If the mass of the board is nine times that of the particle, find the angular velocities of the string and board when *AP* and *AD* are in the same straight line.

21. A heavy circular disc is supported from a point whose height above it is equal to the radius by three equal strings attached to the circumference at equal intervals. One of the strings being cut, shew that the tension of the others is immediately diminished in the ratio of 2 : 3.

22. If one end of a rigid rod of length $6a$ be made to move in the circumference of a circle of radius a with the uniform angular velocity ω round the centre; shew that at the time t, the angle between the rod and the radius of the circle to that end is given by the equation

$$\cot\frac{\phi}{4} = \epsilon^{\frac{\omega t}{2}},$$

the rod being initially in a radius produced.

23. Two rods *AB*, *BC* connected by a hinge at *B* are in motion on a smooth horizontal plane, the end *A* being fixed. If, initially, *AB* has no angular velocity, that of *BC* being ω, shew that when *BC* has no angular velocity that of *AB* will be $b\omega/2a$ and the angle between the rods will then be $\cos^{-1}\{2(2b-a)/3b\}$, $2a$ and $2b$ being the lengths of the rods, which are supposed equal in mass.

24. A sphere falling vertically impinges on a fixed inclined plane, the sphere having an angular velocity ω about

a horizontal axis parallel to the plane. If it be reflected vertically, shew that its angular velocity is

$$(5e^2 - 9e)\,\omega/(5 - 9e),$$

e being the modulus of elasticity.

25. A smooth spherical shell of mass M rests on an inclined plane, being fastened to a point of the plane by a string; a particle m rests inside the sphere; prove that if the string be cut, the ratio of the initial pressure between the sphere and the particle to that between the sphere and the plane is

$$\frac{m\,(1 - \sin\alpha\cos\alpha)}{(M + m)\cos\alpha},$$

where α is the inclination of the plane to the horizon.

26. The ends of a straight rod are moveable on two smooth fixed rods, intersecting each other at right angles; if the rod be set in motion, prove that when θ is its inclination to either fixed rod, the measure of the tendency to break at any point is proportional to $\sin 2\theta$.

27. Two equal rods AB, AC are jointed together and rest symmetrically over a smooth sphere; the junction of the rods at A being severed, what is the initial pressure of each rod on the sphere?

Suppose the sphere divided by a vertical plane through A, perpendicular to the plane of the rods, and imagine the left-hand hemisphere to be suddenly annihilated; it is required to determine the initial action at A.

28. P and Q are two points in a uniform rod equidistant from its centre. The rod can move freely about a hinge at P. The hinge is constrained to move up and down in a vertical line. If the motion be such that Q moves in a horizontal line, determine the velocity when the rod has any given inclination, the rod being supposed to start from rest in a horizontal position.

In the case in which the whole length of the rod $= \sqrt{3} \,.\, PQ$, shew that the time of a complete oscillation

$$= (2\pi)^{\frac{3}{2}} \left(\Gamma \tfrac{1}{4}\right)^{-2} \sqrt{PQ/2g}.$$

In this case also find the equation to the hodograph of the middle point of the rod:

29. A number (n) of equal uniform rods, AA_1, A_1A_2, A_2A_3, &c., are jointed together at their ends A_1, A_2,... and the end A of the first rod is attached to a fixed point. The rods are held so as to form a straight line $AA_1 \ldots A_n$, the end A_n being free, and the supports are simultaneously removed. Prove that if w be the weight of a rod, and $\alpha = 15^\circ$, the initial action at A_r is equal to

$$\frac{w(-4)^r}{2\sqrt{(3)}} \cdot \frac{\sin^{2r}\alpha \cos^{2n}\alpha - \cos^{2r}\alpha \,.\, \sin^{2n}\alpha}{\sin^{2n}\alpha + \cos^{2n}\alpha}.$$

30. AB, BC, CD are three equal uniform rods freely jointed together and moveable about the extremity A; the rods fall from a horizontal position of rest: prove that the radius of curvature of the initial path of the extremity D of the further rod is $\frac{81}{131} a$, where a is the length of each rod. Prove also that the initial stresses at C, B and A are in the ratio of 1, 4 and 15.

31. An arc of a circle is placed in its position of equilibrium in a vertical plane resting on a perfectly rough horizontal plane and slightly disturbed in the former plane; shew that the square of the time of oscillation varies as $a\alpha \sin\alpha$, a being the radius of the circle, and 2α the angle subtended at its centre by the arc.

32. A ball is projected from a point in a perfectly rough horizontal plane, without any rotation; if the coefficient of frictional elasticity be 2/5, prove that the horizontal velocity of the ball, after the n^{th} rebound, will be $\frac{5u}{7} + \left(-\frac{2}{5}\right)^n \frac{2u}{7}$, where u is the horizontal velocity of projection.

33. Two equal rods, connected by a hinge, which allows them to move in a vertical plane, rotate uniformly about a vertical axis through the hinge; and a string, whose length

is double that of either rod, is fastened to their extremities, and supports a weight at its middle point. Determine the angular velocity when in the position of relative equilibrium the rods and the string form a square; and supposing the weight slightly displaced in a vertical direction, find the time of a small oscillation.

34. A smooth plane of mass M is freely moveable about a horizontal axis lying within it and passing through its centre of gravity, the radius of gyration of the plane about the axis being k. The plane being inclined at an angle α to the vertical, a sphere of mass m is placed gently upon it. If initially the centre of the sphere be in a vertical through the axis of the plane, and if h be its initial height above that axis, shew that the angle ϕ, which the initial direction of motion of the centre makes with the vertical, is given by

$$(Mk^2 + mh^2) \tan \phi = Mk^2 \tan \alpha.$$

35. A uniform beam of mass M and length $2a$ can turn round a fixed horizontal axis at one end; to the other end of the beam a string of length l is attached, and at the end of the string is a particle of mass m. Determine the relation that must hold in order that, during a small oscillation of the system, the inclination of the string to the vertical may be twice that of the beam.

36. If a given power P raise a given weight Q through a given space in the least possible time, on a wheel and axle, the masses of which are M and M' respectively, prove that the ratio of their radii must be

$$\frac{Q}{P} + \sqrt{\frac{Q^2}{P^2} + \frac{2Q + M'}{2P + M'}} \; : \; 1.$$

37. A uniform heavy beam of length $2c$ is supported in a horizontal position by means of two strings, without weight, each of length b, which are fastened to its ends, the other ends of the strings being fixed; in equilibrium each of the strings makes an angle α with the horizon: find the time of a small oscillation when the system is slightly displaced in the vertical plane in which it is situated, the strings not being slackened.

38. A circular ring hangs in a vertical plane on two pegs. If one be removed, prove that, P_1, P_2 being the instantaneous pressures on the other peg, calculated on the supposition that the ring is (1) smooth, (2) rough,

$$P_1^2 : P_2^2 :: 4 : 4 + \tan^2\alpha,$$

where α is the angle which the line drawn from the centre of the ring to the peg makes with the vertical.

39. A heavy bar is suspended in a horizontal position by two equal and parallel vertical strings attached to its ends, it is then set swinging so that the strings move in vertical planes perpendicular to the bar, if one string breaks when the rod is in its lowest position, prove that the tension of the other string is instantaneously diminished by one half its value.

If the second string be cut when the bar is vertical, prove that the subsequent rotation will be uniform and round a horizontal axis fixed in direction; but if the second string be cut at any other time, the vertical plane containing the bar will rotate with an angular velocity varying as $\sec^2\theta$, and θ will increase at a rate varying as $\sqrt{a + b\sec^2\theta}$, where θ is the inclination to the horizon and a, b are constant.

40. A chain of mass m and length l hangs in equilibrium over a smooth pulley, an insect of mass M alights gently at one end and begins crawling up with uniform relative velocity V; shew that the velocity with which the chain leaves the pulley will be

$$\left\{\frac{M^2}{(M+m)^2}V^2 + \frac{M + \frac{1}{2}m}{M + m}gl\right\}^{\frac{1}{2}}.$$

41. A man walks on a large rough ball so as to make the ball roll straight up an inclined plane of inclination α, keeping himself at an angular distance β from the highest point of the ball; prove that the acceleration of the man is

$$5g\frac{\sin\beta - 2\sin\alpha}{7 + 10\cos^2\frac{1}{2}(\alpha+\beta)},$$

taking the man and ball to have equal weights.

42. A light uniform lamina in the form of a regular trapezoid is suspended by one of the parallel edges, and a weight mg is uniformly distributed over the opposite edge; supposing the lamina to be elastic only in the direction of the breadth, find the position of equilibrium.

Shew that the time of a small oscillation is

$$2\pi\sqrt{\frac{ml\,(\log a - \log b)}{2\lambda\,(a-b)}},$$

when $2a$ and $2b$ are the lengths of the parallel edges, l is the breadth of the lamina when unstretched, and λ the modulus of elasticity.

43. The extremities of a uniform heavy rod of length $2c$ slide on a smooth wire in the form of the parabola $x^2-4ay=0$, the axis of the parabola being vertical, and $c>2a$. If the rod be slightly displaced from its position of stable equilibrium, prove that the time of a small oscillation is

$$2\pi\left\{\frac{2ac}{3g\,(c-2a)}\right\}^{\frac{1}{2}}.$$

44. Four equal uniform rods are jointed together so as to form a square $ABCD$, and the system is suspended from the joint A, the square form being maintained by an elastic string joining A and C.

Find the tension of the string, and, the modulus of elasticity being twice the weight of one of the rods, prove that, if C be slightly depressed, the length of the simple isochronous pendulum will be $5AC/12$. If, when there is equilibrium, the string be cut, prove that the initial pressure at C is equal to one-tenth of the weight of one of the rods, and that the initial acceleration of C is equal to $6g/5$.

45. A cylindrical mass of snow rolls down an inclined plane covered with snow of uniform depth e; gathering up all the snow it rolls over, and always remaining circular. Prove that it will move with uniform acceleration $g\sin\alpha/5$ if initially when its radius is a it be started with velocity $a\sqrt{2\pi g\sin\alpha/5e}$, where α is the inclination of the plane to the horizon.

46. Three particles A, B, C are connected by two strings AB, AC and placed in a line on a smooth table. The extreme particles B and C are then projected at right angles to the strings with velocities u, v. Prove that the initial curvatures of the paths of the extreme particles are respectively

$$\frac{(q+m)\frac{u^2}{a}+q\frac{v^2}{b}}{(p+q+m)u^2} \text{ and } \frac{(p+m)\frac{v^2}{b}+p\frac{u^2}{a}}{(p+q+m)v^2},$$

m, p, q being the masses of the particles, and a, b the lengths of the strings.

47. Two particles of masses m_1, m_2 are tied to the extremities of a string that passes through a bead of mass M, the whole is placed on a smooth table with m_1, m_2 at the acute angles and M at the right angle of a right-angled triangle, if the particles at the ends be projected with velocities u_1, u_2, perpendicular to the respective strings, then shew that the initial radii of curvature of their paths are ρ_1, ρ_2

where $$\frac{m_1u_1^2}{\rho_1}=\frac{m_2u_2^2}{\rho_2}=\frac{\frac{u_1^2}{a_1}+\frac{u_2^2}{a_2}}{\frac{1}{m_1}+\frac{2}{M}+\frac{1}{m_2}}.$$

48. A circular disc of mass M, radius a, and moment of inertia about the centre MK^2, is spinning with angular velocity Ω and impinges normally with any velocity on a rough rod resting on a smooth plane. Shew that the angular velocity immediately after impact becomes reduced to

$$\frac{(M+m)K^2}{ma^2+(M+m)K^2}\Omega.$$

49. Two uniform rods AB, BC of masses m, m' freely jointed at B lie upon a smooth horizontal table and AB is struck perpendicular to its length at a point between A and B; shew that the point B will begin to move in a direction making with BC an angle $\tan^{-1}\left\{\frac{4(m+m')}{4m+m'}\cot\alpha\right\}$; α being the angle between the rods.

50. A spherical shell of mass m, whose outer surface is rough and of radius a, has its inner surface smooth and of radius b; a particle of mass m moves inside while the shell rolls on a rough table, shew that if the excursions of the particle be α on either side of the vertical, then

$$[M(a^2+k^2)+ma^2\sin^2\theta]\,\dot{\theta}^2$$
$$=\frac{2g}{b}[M(a^2+k^2)+ma^2][\cos\theta-\cos\alpha].$$

51. A smooth massless rod HM of length $l+2a$ turns freely about a hinge at one end H. A string of length l is fastened at the end M and also at a point S in a horizontal line with H and at a distance $2a\sqrt{2}$ from H. A smooth ring of mass m is slipped over the rod and string at M and moved up the rod until the string is tight and the rod horizontal; it is then allowed to fall, find the velocity of the ring at any instant before it slips off the rod. Shew that the tension of the string when the ring has fallen through a vertical height y is

$$T=\frac{mg}{2a}\cdot\frac{3a^2+2y^2}{a^2+2y^2}\cdot y,$$

mg being the weight of the ring.

52. A smooth thin spherical shell of mass M and radius a rests on a smooth inclined plane by means of an elastic string which is attached to the sphere and to a peg at the same distance from the plane as the centre of the sphere and a particle of mass m rests on the inner surface of the shell. In the position of equilibrium the string is parallel to the plane, find the times of oscillation of the system when it is slightly displaced in a vertical plane and prove that the arc traversed by the particle and the distance traversed by the centre of the shell from their positions of equilibrium can always be equal if

$$Mg+mg(1+\cos\alpha)=\lambda a(1+\cos\alpha)/c,$$

where λ is the coefficient of elasticity of the string, and c its natural length.

53. A uniform circular disc moving in any way is placed gently upon a rough horizontal plane. Assuming that the

friction between any element of the disc and the plane varies as the relative velocity and is in a direction opposite to it, find the motion of the disc, and shew that if u and ω be the velocity of the centre and the angular velocity about it at any instant, $u\omega_0 = u_0\omega$, where u_0 and ω_0 are the initial values of u and ω.

54. A number (n) of equal uniform rods $A_1C_1B_1$, $A_2C_2B_2$,... are placed on a smooth horizontal plane so that the end A_2 of the 2nd is in contact with the middle point C_1 of the first, the end A_3 in contact with C_2... and the angles $C_2A_2B_1$, $C_3A_3B_2$... are each equal to θ, so that the figure $A_1C_1C_2C_3 \ldots C_n$ is a portion of a regular polygon. At the end A_1 an impulse P is applied inwards in the direction making an angle $\pi/2 - \theta$ with A_1C_1. Prove that the impulse between the r^{th} and $\overline{r+1}|^{\text{th}}$, supposing them smooth and rigid,

$$= P\,(\beta^n\alpha^r - \alpha^n\beta^r)/(\beta^n - \alpha^n),$$

where α and β are the roots of the equation

$$z^2 - (2\sec\theta + 3\cos\theta)\,z + 1 = 0.$$

CHAPTER XIV.

MOTION IN THREE DIMENSIONS.

230. WE now proceed to consider the motion of a system referred to three rectangular axes, either fixed, or moving in a given manner.

As in Art. (28) we employ θ_1, θ_2, and θ_3 to represent the angular velocities of the system of axes.

Taking ω_1, ω_2, and ω_3 as the angular velocities, at any instant, of a rigid body about the axes, it follows as in Art. (28) that the angular accelerations are respectively

$$\dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2,$$
$$\dot{\omega}_2 - \omega_3\theta_1 + \omega_1\theta_3,$$
$$\dot{\omega}_3 - \omega_1\theta_2 + \omega_2\theta_1.$$

From the definition of the linear momenta and the angular momenta of a system it follows that these quantities are vectors and are subject to the parallelogrammic law.

Let p_1, p_2, p_3 represent the linear momenta of a system in the directions of the axes, and h_1, h_2, h_3 the angular momenta of the system about those axes.

Then it follows, as in Art. 28, that, if we take *OL*, *OM*, and *ON* to represent either the quantities p_1, p_2, p_3 or the quantities h_1, h_2, h_3, the rates of change of these quantities are, on the same scale, the velocities of the point of which *OL*, *OM*, *ON* are co-ordinates, and are therefore respectively

$$\left.\begin{matrix}\dot{p}_1 - p_2\theta_3 + p_3\theta_2 \\ \dot{p}_2 - p_3\theta_1 + p_1\theta_3 \\ \dot{p}_3 - p_1\theta_2 + p_2\theta_1\end{matrix}\right\} \text{ and } \left.\begin{matrix}\dot{h}_1 - h_2\theta_3 + h_3\theta_2 \\ \dot{h}_2 - h_3\theta_1 + h_1\theta_3 \\ \dot{h}_3 - h_1\theta_2 + h_2\theta_1\end{matrix}\right\}.$$

The equations of motion of the system are obtained by equating these expressions to the components of the acting forces and of the acting couples.

The equations of motion, in the forms thus obtained, were first given by Mr R. B. Hayward, F.R.S., of St John's College, Cambridge.

They are contained in a paper published in 1856, in Part I., Vol. X., of the *Cambridge Philosophical Transactions.*

231. If x, y, z be the co-ordinates of a particle m of the system, and if u, v, w be the component velocities of the particle,

$$p_1 = \Sigma mu,\ p_2 = \Sigma mv,\ p_3 = \Sigma mw;$$
$$h_1 = \Sigma m\,(wy - vz),\ h_2 = \Sigma m\,(uz - wx),\ h_3 = \Sigma m\,(vx - uy).$$

The total motion of the system at the instant in question is thus represented by three linear momenta in the directions of the axes and three angular momenta about those axes.

These are equivalent to a single linear momentum and a single angular momentum.

232. If the origin be not a fixed point, the expressions for the rates of change of the linear momenta are unaffected, but the expressions for the rates of change of angular momenta will require modification.

Let α, β, γ be the component velocities of the origin, and suppose the axes to have no rotation.

Since $$h_1 = \Sigma m\,(wy - vz),$$

the angular momentum, at the time $t + \delta t$, about the axis Ox, fixed in space

$$= \Sigma m\,\{(w + \delta w)\,(y + \delta y + \beta\delta t) - (v + \delta v)\,(z + \delta z + \gamma\delta t)\}$$
$$= h_1 + \delta h_1 + (p_3\beta - p_2\gamma)\,\delta t,$$

and, subtracting h_1 and dividing by δt, we obtain the additional term

$$p_3\beta - p_2\gamma,$$

so that the complete expression for the time-flux, about the instantaneous position of the axis, of the angular momentum is

$$\dot{h}_1 - h_2\theta_3 + h_3\theta_2 + p_3\beta - p_2\gamma.$$

If the origin be the centre of gravity of the system, the expressions for angular momenta and their rates of change are those of Art. (230).

It will be seen that the terms $p_3\beta - p_2\gamma$ of the previous article disappear in this case, for

$$p_2 = M\beta, \text{ and } p_3 = M\gamma.$$

233. *Motion of a rigid body about a fixed point.*

In this case

$$u = z\omega_2 - y\omega_3,\quad v = x\omega_3 - z\omega_1,\quad w = y\omega_1 - x\omega_2,$$

and therefore

$$h_1 = \Sigma m\,(y^2 + z^2)\,\omega_1 - \Sigma\,(mxy)\,\omega_2 - \Sigma\,(mxy)\,\omega_3,$$

and, if we represent the three moments of inertia by A, B, C and the three products of inertia by D, E, F we have

$$h_1 = A\omega_1 - F\omega_2 - E\omega_3,$$
$$h_2 = B\omega_2 - D\omega_3 - F\omega_1,$$
$$h_3 = C\omega_3 - E\omega_1 - D\omega_2.$$

If the expressions D, E, F all vanish the axes are said to be principal axes; if two vanish, the corresponding axis is a principal axis.

In the case of a sphere, or a solid bounded by any regular polyhedron, when the centre is the origin, D, E, F all vanish, and A, B, C are all the same, so that the angular momenta take the forms

$$A\omega_1,\ A\omega_2,\ A\omega_3.$$

In the case of a solid of revolution, or of a regular pyramid, the axis of which is one of the axes, D, E, F all vanish, and the angular momenta are

$$A\omega_1,\ A\omega_2,\ C\omega_3.$$

In the case of a plane lamina, when one axis is perpendicular to its plane, D and E vanish, and the angular momenta are

$$A\omega_1 - F\omega_2,\ B\omega_2 - F\omega_1,\ (A+B)\,\omega_3.$$

234. If the axis of z be fixed in space, the time-fluxes of angular momenta about the instantaneous positions of the axes are

$$\dot{h}_1 - h_2\theta_3,\ \dot{h}_2 + h_1\theta_3,\ \dot{h}_3.$$

It may be instructive to obtain these expressions directly. Thus, at the time $t + \delta t$, the angular momenta about Ox', Oy', the consecutive positions of Ox, Oy, being $h_1 + \delta h_1$, $h_2 + \delta h_2$, it follows that the angular momenta about Ox and Oy are respectively

$$(h_1 + \delta h_1)\cos\theta_3\delta t - (h_2 + \delta h_2)\sin\theta_3\delta t,$$
$$(h_2 + \delta h_2)\cos\theta_3\delta t + (h_1 + \delta h_1)\sin\theta_3\delta t,$$

and, subtracting h_1 and h_2 and dividing by δt, we obtain, in the limit, the expressions given above.

The general expressions of Art. 230 may be obtained in a similar manner.

235. We are now in a position to solve some problems, and we commence with the *motion of a sphere on a rough plane, under the action of forces the resultant of which passes through the centre of the sphere.*

Referring to fixed axes the linear momenta are $m\dot{x}$, $m\dot{y}$, and the time-fluxes of the angular momenta are

$$A\dot{\omega}_1,\ A\dot{\omega}_2,\ A\dot{\omega}_3 \text{ where } A = 2mc^2/5.$$

Assuming X, Y as the forces, and taking moments about the lines in the plane through the point of contact parallel to x and y, we obtain

$$-m\ddot{y}c + A\dot{\omega}_1 = -Yc, \quad m\ddot{x}c + A\dot{\omega}_2 = Xc.$$

We have also the geometrical conditions,

$$\dot{x} - c\omega_2 = 0, \quad \dot{y} + c\omega_1 = 0,$$

and we hence obtain

$$m\ddot{x} = \tfrac{5}{7}X, \quad m\ddot{y} = \tfrac{5}{7}Y.$$

If the frictional reactions be required, they are given by the equations

$$m\ddot{x} = F + X, \quad m\ddot{y} = G + Y,$$

so that $$F = -\tfrac{2}{7}X \text{ and } G = -\tfrac{2}{7}Y.$$

If the plane be made to revolve uniformly, with the angular velocity Ω about the axis of z, the equations of motion are the same, but the geometrical conditions are

$$\dot{x} - c\omega_2 = -\Omega y, \quad \dot{y} + c\omega_1 = \Omega x.$$

If in this case there be no forces in action, the elimination of ω_1 and ω_2 leads to the equations

$$7\ddot{x} + 2\Omega\dot{y} = 0, \quad 7\ddot{y} - 2\Omega\dot{x} = 0,$$

or, if $$7n = 2\Omega, \quad \ddot{x} + n^2(x - a) = 0, \quad + n^2(y - b) = 0,$$

where a and b are constants.

Integrating these equations and eliminating the time, we shall find that the path of the centre of the sphere is an ellipse, of which the point (a, b) is the centre.

236. To illustrate the use of two moving axes, consider the motion of a rigid body about a fixed axis.

Taking, as in Art. 211, the line OG as the axis of x and the fixed axis as the axis of z,

$$u = -y\omega, \quad v = x\omega, \quad w = 0,$$

and therefore,

$$h_1 = -E\omega, \quad h_2 = -D\omega, \quad h_3 = C\omega.$$

Hence
$$\dot{h}_1 - h_2\theta_3 = -E\dot{\omega} + D\omega^2,$$
$$\dot{h}_2 + h_1\theta_3 = -D\omega - E\omega^2,$$
and equating these expressions to the moments of the acting forces, we obtain, as in Art. 211, the stresses on the axis.

237. *A circular disc, the plane of which is vertical, and centre fixed, is rotating about a horizontal axis through its centre perpendicular to its plane, which axis is itself rotating freely in the horizontal plane through the centre, and an insect crawls in a given manner on the disc.*

Taking the figure of Art. (247), let ZC be the plane of disc, OC being a given radius of the disc.

The equations of motion are obtained by observing that the angular momentum about Oz is constant, and that the time-flux of the angular momentum about OF is equal to the moment about OF of the weight of the insect.

Let ρ be the distance of the insect from O, and ϕ the angular distance of ρ from OC measured in the direction CE, so that ρ and ϕ are known functions of the time.

Putting $ZC = \theta$, and $XZC = \psi$, we obtain
$$Mk^2\dot{\psi} + m\rho^2 \sin^2(\theta + \phi)\,\dot{\psi} = C.$$

If h_1, h_2 be the angular momenta about OK and OF,
$$h_1 = m\rho^2\dot{\psi} \sin(\theta + \phi) \cos(\theta + \phi),$$
$$h_2 = 2Mk^2\dot{\theta} + m\rho^2(\dot{\theta} + \dot{\phi}).$$

Hence we obtain, since $\dot{h}_2 + h_1\dot{\psi}$ is the time-flux of the angular momentum about OF,
$$2Mk^2\ddot{\theta} + m\rho^2(\ddot{\theta} + \ddot{\phi}) + 2m\rho\dot{\rho}(\dot{\theta} + \dot{\phi}) + m\rho^2\dot{\psi}^2 \sin(\theta + \phi)\cos(\theta + \phi)$$
$$= mg\rho \sin(\theta + \phi),$$
and θ and ψ are determined by these equations.

238. *Motion of a heavy sphere on the interior rough surface of a vertical cylinder.*

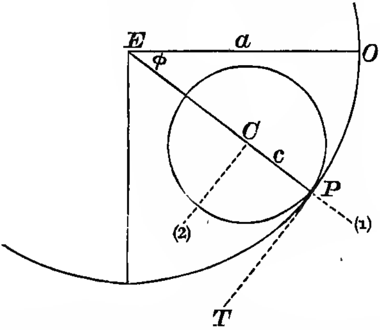
The figure being a section by the horizontal plane through the centre of the sphere, take the axis (3) through C vertically upwards and measure z upwards.

The accelerations of C in the directions (1) (2) and (3) are

$$-(a-c)\dot{\phi}^2,\ (a-c)\ddot{\phi},\ \ddot{z}.$$

In this case, since $h_1 = A\omega_1$, $h_2 = A\omega_2$, and $\theta_3 = \dot{\phi}$, the time-fluxes of the angular momenta about the instantaneous positions of the lines (1), (2), and (3) are

$$A\dot{\omega}_1 - A\omega_2\dot{\phi},\ A\dot{\omega}_2 + A\omega_1\dot{\phi},\ A\dot{\omega}_3.$$



Taking moments about CP, PT, and the vertical through P, we obtain

$$\dot{\omega}_1 - \omega_2\dot{\phi} = 0,\ A\dot{\omega}_2 + A\omega_1\dot{\phi} + mc\ddot{z} = -mgc,\ A\dot{\omega}_3 - m(a-c)c\ddot{\phi} = 0.$$

Expressing the fact that the point P of the sphere has no velocity, the geometrical conditions are

$$(a-c)\dot{\phi} + c\omega_3 = 0,\ \ \dot{z} - c\omega_2 = 0,$$

we at once see that $\dot{\phi}$ and ω_3 are constant, that $c\dot{\omega}_1 = \dot{\phi}\dot{z}$, and therefore that $A\ddot{z} + A\dot{\phi}^2 z + mc^2\ddot{z}$ is constant, or that if $2\dot{\phi}^2 = 7n^2$,

$$\ddot{z} + n^2 z = C,$$

shewing that the ball rolls up and down, between two fixed levels, in the time π/n.

If the cylinder, instead of being fixed, be made to revolve with a constant angular velocity ω about a vertical generating line through the point O in the figure, the angular velocity of the line CE is $\omega + \dot{\phi}$, and the accelerations of C in the directions (1), (2) and (3) are, putting b for $a - c$,

$$a\omega^2\cos\phi - b(\omega + \dot{\phi})^2,\ b\ddot{\phi} - a\omega^2\sin\phi,\ \ddot{z}.$$

Hence, taking moments about the same axes as before, we obtain,

$$\dot\omega_1 - \omega_2(\omega + \dot\phi) = 0,\ \tfrac{2}{5}c\dot\omega_2 + \tfrac{2}{5}c\omega_1(\omega + \dot\phi) + \ddot z = -g,$$

$$\tfrac{2}{5}c\dot\omega_3 - b\ddot\phi + a\omega^2\sin\phi = 0.$$

The geometrical conditions are that the velocities of the point P of the sphere and of the point P of the cylinder are the same,

$$c\omega_3 + b(\omega + \dot\phi) - a\omega\cos\phi = a\omega - a\omega\cos\phi$$

or

$$c\omega_3 + b(\omega + \dot\phi) = a\omega,$$

and

$$\dot z - c\omega_2 = 0.$$

Eliminating ω_3 we find that

$$7b\ddot\phi = 5a\omega^2\sin\phi,$$

an equation which determines the angular motion of the centre of the sphere relative to the cylinder.

239. *Motion of a heavy sphere on the interior rough surface of a cone having its axis vertical and vertex downwards.*

The figure being a section of the system by the vertical plane through the axis of the cone and the centre of the sphere, the accelerations of C are

$$\ddot r - r\dot\phi^2,\quad r\ddot\phi + 2\dot r\dot\phi,\quad \ddot z,$$

and the time-fluxes of angular momenta are the same as in the preceding case.

The geometrical conditions are

$$\dot r - c\omega_2\sin\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$r\dot\phi + c\omega_1\sin\alpha + c\omega_3\cos\alpha = 0 \ldots\ldots\ldots\ldots(2)$$

$$\dot z - c\omega_2\cos\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

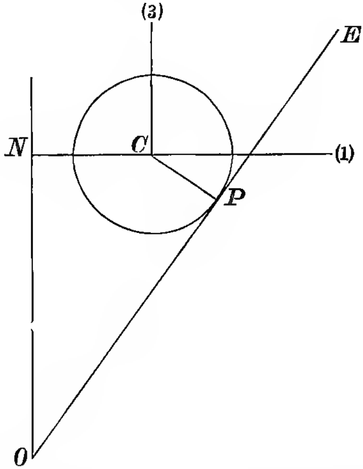
Taking moments about PE, PC, and the line through P perpendicular to the plane of the figure we obtain

$$(A\dot\omega_1 - A\omega_2\dot\phi)\sin\alpha + A\dot\omega_3\cos\alpha - m(r\ddot\phi + 2\dot r\dot\phi)c = 0 \ldots(4)$$

$$(A\dot\omega_1 - A\omega_2\dot\phi)\cos\alpha - A\dot\omega_3\sin\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$A\dot\omega_2 + A\omega_1\dot\phi + m(\ddot r - r\dot\phi^2)c\sin\alpha + m\ddot z c\cos\alpha = -mgc\cos\alpha \ldots(6).$$

From (4) and (2) we find that $r\ddot{\phi}+2\dot{r}\dot{\phi}=0$, or that $r^2\dot{\phi}=h$,
From (4) and (5) it follows that $\dot{\omega}_3=0$, or that $\omega_3=n$.



Lastly from (6), with the aid of (1), (2), (3), and the preceding results, we find, if we write u for $1/r$, the equation,

$$\frac{d^2u}{d\phi^2}+\frac{2+5\sin^2\alpha}{7}u=\tfrac{5}{7}\frac{g\sin\alpha\cos\alpha}{h^2u^2}-\tfrac{2}{7}\frac{cn\cos\alpha}{h}.$$

240. The general problem of the motion of a sphere on any surface of revolution may be treated in the same manner, or we may employ three moving axes.

Taking the axes as in the figure, and taking u, v, w as the velocities of C in the directions (1), (2), and (3), the geometrical conditions are

$$u-c\omega_2=0,\ \ v+c\omega_1=0,\ \ w=0.$$

Since $\theta_1=\dot{\phi}\cos\theta,\ \theta_2=\dot{\theta},\ \theta_3=\dot{\phi}\sin\theta$,

$$\begin{aligned}\dot{u}-v\theta_3+w\theta_2&=c\dot{\omega}_2+c\omega_1\dot{\phi}\sin\theta,\\ \dot{v}-w\theta_1+u\theta_3&=-c\dot{\omega}_1+c\omega_2\dot{\phi}\sin\theta,\\ \dot{w}-u\theta_2+v\theta_3&=-c\omega_2\dot{\theta}-c\omega_1\dot{\phi}\cos\theta,\end{aligned}$$

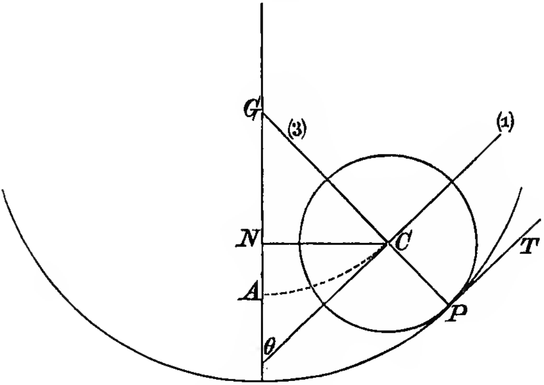
which are the accelerations of C.

Hence, if L, M, N be the moments of the acting forces about PT, about the line through P perpendicular to the plane AGP, and about PC, the equations of motion are

$$A(\dot{\omega}_1 - \omega_2\dot{\phi}\sin\theta + \omega_3\dot{\theta}) - mc(-c\dot{\omega}_1 + c\omega_2\dot{\phi}\sin\theta) = L, \ldots \text{i},$$

$$A(\dot{\omega}_2 - \omega_3\dot{\phi}\cos\theta + \omega_1\dot{\phi}\sin\theta) + mc(c\dot{\omega}_2 + c\dot{\omega}_1\dot{\phi}\sin\theta) = M, \ldots \text{ii},$$

$$A(\dot{\omega}_3 - \omega_1\dot{\theta} + \omega_2\dot{\phi}\cos\theta) = N \ldots \text{iii}.$$



It will be seen that C moves on a parallel surface of which A is the vertex, and if $AN = z$ and $CN = r$, the relation between r and z is known, so that $z = f(r)$.

Hence if s be the arc AC

$$c\omega_2 = u = s = -\rho\dot{\theta},$$

if ρ be the radius of curvature of AC at C.

We also have $$c\omega_1 = -v = -r\dot{\phi}.$$

If for example the surface be a sphere, and gravity the only force in action,

$$L = 0,\ N = 0, \text{ and } M = -mgc\cos\theta.$$

Also, in this case,

$$c\omega_2 = -a\dot{\theta},\ c\omega_1 = -a\cos\theta\dot{\phi},$$

and therefore, from equation iii,

$$\dot{\omega}_3 = 0, \text{ or } \omega_3 = n.$$

These values of ω_1, ω_2, ω_3 being substituted in equation i, we obtain

$$a\cos\theta\ddot{\phi} - 2a\sin\theta\,\dot{\theta}\dot{\phi} = \tfrac{2}{7}cn\dot{\theta},$$

and therefore

$$\cos^2\theta\dot{\phi} = C + \tfrac{2}{7}\frac{cn}{a}\sin\theta.$$

Substituting in equation ii and integrating we obtain a differential equation of the first order for θ.

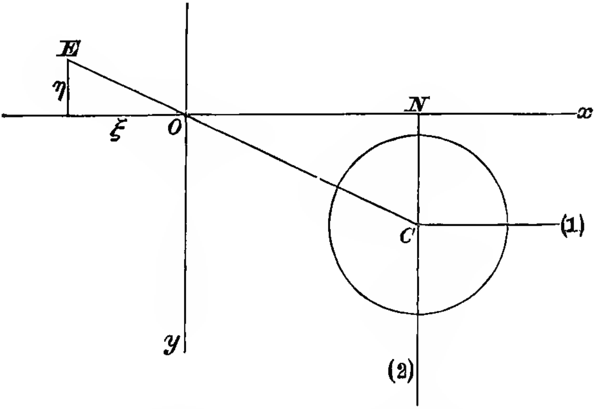
This last however is more easily obtained from the equation of energy which is

$$a^2\dot{\theta}^2 + a^2\cos^2\theta\dot{\phi}^2 + \tfrac{2}{5}c^2(\omega_1^2 + \omega_2^2 + \omega_3^2) = D + 2ga\cos\theta,$$

or $$\dot{\theta}^2 + \sec^2\theta\left(C + \tfrac{2}{7}\frac{cn}{a}\sin\theta\right)^2 = E + \frac{10g}{7a}\cos\theta,$$

C and E being constants determined by initial conditions.

241. *Motion of a rough sphere on the surface of a flat disc, which is moveable on a smooth horizontal plane, the upper surface of the disc being perfectly rough.*



We shall suppose that the centre of gravity of the system has no motion; this will be the case if the disc be initially at rest, and if the sphere, in a state of rotation about a diameter, be placed gently upon the disc.

In the figure O is the projection on a horizontal plane of the centre of gravity of the system, E of the centre of gravity of the disc, and C of the centre of the sphere; Ox, Oy are fixed directions.

Taking ξ, η, and x, y, as the co-ordinates of E and O, measured in opposite directions, and Ω as the angular velocity of the disc, the geometrical conditions are

$$\dot{x} - c\omega_2 = -\dot{\xi} - \Omega(y+\eta), \text{ ....}(1)$$

$$\dot{y} + c\omega_1 = -\dot{\eta} + \Omega(x+\xi). \text{ ....}(2).$$

Taking moments for the motion of the sphere, about the horizontal tangent lines parallel to the axes,

$$m\ddot{x}c + mk^2\dot{\omega}_2 = 0, \quad m\ddot{y}c - mk^2\dot{\omega}_1 = 0,$$

and therefore $\dot{x} + \frac{2}{5}c\omega_2 = A, \quad \dot{y} - \frac{2}{5}c\omega_1 = B$......(3), (4).

The angular momentum of the system about any assigned vertical line is constant, and if the sphere have initially no rotation about the vertical diameter this constant is zero.

Taking moments about the vertical line, fixed in space, through which E is passing,

$$MK^2\Omega = m\dot{x}(y+\eta) - m\dot{y}(x+\xi),$$

or $$M^2K^2\Omega = m(m+M)(\dot{x}y - \dot{y}x)\text{...............}(5).$$

We have besides $mx = M\xi$, and $my = M\eta$, and we thus have seven equations to determine the seven unknown quantities.

If the original axis of rotation of the sphere be above the line Ox and parallel to it, so that initially

$$\omega_1 = n, \text{ and } \omega_2 = 0,$$

we obtain $$\dot{x} + \frac{2}{5}c\omega_2 = 0, \quad \dot{y} - \frac{2}{5}c\omega_1 = B.$$

The elimination of ω_1 and ω_2 leads to

$$2\dot{x}(M+m) + 5M\dot{x} = -2y\Omega(M+m),$$

$$2\dot{y}(M+m) + 5M\dot{y} - 5MB = 2x\Omega(M+m),$$

and, substituting for Ω its value from (5), these equations take the forms

$$(a+by^2)\,\dot{x} = bxy\,\dot{y},\quad (a+bx^2)\,\dot{y} = bxy\,\dot{x} + c,$$

where a, b and c are constants.

The integration of the first of these equations gives

$$a + by^2 = Cx^2,$$

shewing that the path in space of the centre of the sphere is a hyperbola, (a result given in the Tripos Examination, Jan. 1882).

It should be mentioned that n is the initial angular velocity of the sphere just after having been placed in contact with the disc.

If ω be the angular velocity of the sphere about the diameter parallel to the axis of x before the contact, and if λ be the angular velocity of the disc immediately after the impact, n and λ are determined by the equation

$$mk^2n - m\dot{y}_0c = mk^2\omega,$$

combined with the preceding equations (1), (2), and (5) in their initial forms.

242. *Motion of a heavy rod AB, the ends of which slide on a fixed vertical rod OB, and a horizontal rod OA, which is made to revolve uniformly.*

If r be the distance from G, in the direction GA, of a point P of the rod, the accelerations f, f' of the point in the directions LP and NP, are $\frac{d^2}{dt^2}PL - \omega^2 PL$, and $\frac{d^2}{dt^2}PN$, where PL, PN are the perpendiculars upon OB and OA, so that $\quad PL = (a+r)\cos\theta$, and $PN = (a-r)\sin\theta$.

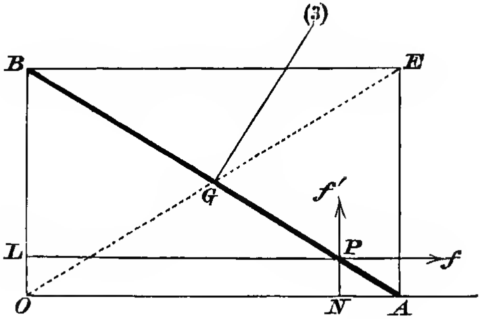
Taking moments about the line through E perpendicular to the plane OAB, we obtain

$$\int_{-a}^{a} m\frac{dr}{2a}\{(a+r)f\sin\theta - (a-r)f'\cos\theta\} = mga\cos\theta.$$

Substituting for f and f' their values, and integrating, this reduces to

$$\ddot{\theta} + \omega^2 \sin\theta \cos\theta = -\frac{3g}{4a} \cos\theta.$$

We have solved this question by an appeal to first principles, but it may be instructive to indicate the method of dealing with it by the aid of the expressions for angular momenta.



Taking for axes the line GA, and the lines through G perpendicular to and in the plane OAB,

$$\theta_1 = -\omega \sin\theta, \quad \theta_2 = \dot{\theta}, \quad \theta_3 = \omega \cos\theta,$$

and $$h_1 = 0, \quad h_2 = mk^2\omega_2, \quad h_3 = mk^2\omega_3.$$

The acceleration of G in the plane OAB, perpendicular to OG $\qquad = a\ddot{\theta} + a \sin\theta \cos\theta \,.\, \omega^2.$ Art. 15,

and the rate of change of the angular momentum about the second axis $\qquad = mk^2\dot{\omega}_2 + mk^2\omega_3\,\omega \sin\theta.$

Hence the equation of moments about the line through E perpendicular to the plane is

$$ma^2(\ddot{\theta} + \sin\theta \cos\theta\omega^2) + m\frac{a^2}{3}\dot{\omega}_2 + m\frac{a^2}{3}\omega_3\omega \sin\theta = -mga\cos\theta;$$

and, observing that $\omega_2 = \dot{\theta}$ and that $\omega_3 = \omega\cos\theta$, this reduces to the equation previously obtained.

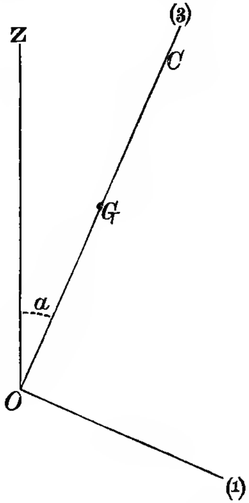
If the system instead of being made to revolve uniformly be set in motion and left to itself, we shall have, taking $\dot{\phi}$ for the azimuthal motion, and neglecting the inertia of the rods OA, OB,

$$\ddot{\theta} + \dot{\phi}^2 \sin\theta \cos\theta = -\frac{3g}{4a}\cos\theta,$$

with the additional equation, derived from the fact that the angular momentum about OB is constant,

$$\int_{-a}^{a} m\frac{dr}{2a}\,(a+r)^2\cos^2\theta \,.\, \dot{\phi} = C.$$

243. *Steady motion of a heavy body in the form of a solid of revolution, rotating uniformly about its axis, one point O of which is fixed while the axis has a constant inclination to the vertical (α), and a constant azimuthal motion (Ω).*



Taking moving axes as in the figure, the second axis being perpendicular to the plane of the paper,

$$\theta_1 = -\Omega\sin\alpha,\ \ \theta_2 = 0,\ \ \theta_3 = \Omega\cos\alpha,$$

also $\omega_2 = 0$ and $\omega_1 = -\Omega \sin \alpha$, and, taking moments about the axes, we obtain

$$A\dot{\omega}_1 = 0,\ C\omega_3 \Omega \sin \alpha + A\omega_1 \Omega \cos \alpha = mga \sin \alpha,\ C\dot{\omega}_3 = 0.$$

Putting n for ω_3 this gives

$$C\Omega n - A\Omega^2 \cos \alpha = mga,$$

as the condition for steady motion.

If $\alpha = \dfrac{\pi}{2}$, $C\Omega n = mga$, so that if the angular velocity n be imparted to the body about OC, when OC is horizontal, and OC be then started with the angular velocity mga/Cn about Oz, the axis OC will continue to revolve in a horizontal plane.

In the general case, when the motion is not steady, take $\dot{\psi}$ as the azimuthal motion, so that

$$\theta_1 = -\dot{\psi} \sin \theta,\ \theta_2 = \dot{\theta},\ \theta_3 = \dot{\psi} \cos \theta,$$

and therefore

$$A\dot{\omega}_1 - A\omega_2\dot{\psi} \cos \theta + C\omega_3\dot{\theta} = 0,$$

$$A\dot{\omega}_2 + C\omega_3\dot{\psi} \sin \theta + A\omega_1\dot{\psi} \cos \theta = mga \sin \theta,$$

$$C\dot{\omega}_3 - A\omega_1\dot{\theta} - A\omega_2\dot{\psi} \sin \theta = 0.$$

Now $\omega_1 = -\dot{\psi} \sin \theta$, and $\omega_2 = \dot{\theta}$, and we find that ω_3 is a constant (n) and that

$$-A\ddot{\psi} \sin \theta - 2A\dot{\psi}\dot{\theta} \cos \theta + Cn\dot{\theta} = 0,$$

$$A\ddot{\theta} + Cn\dot{\psi} \sin \theta - A\dot{\psi}^2 \sin \theta \cos \theta = mga \sin \theta,$$

two equations which completely determine the motion.

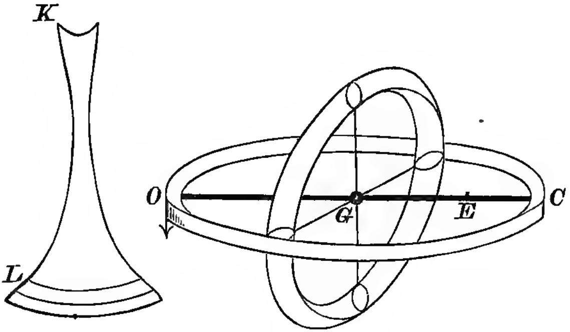
If the motion be very nearly steady, the small oscillations are determined by putting

$$\theta = \alpha + \phi, \text{ and } \dot{\psi} = \Omega + \dot{\chi},$$

and neglecting the squares and products of the small quantities ϕ and $\dot{\chi}$.

These results are roughly illustrated by observing the motion of an ordinary spinning top on a rough horizontal plane.

The most complete illustration is obtained by means of a Gyroscope.



A heavy brass solid ring is moveable about the diameter OC of a circular brass framework, and, holding this framework, a rapid rotation can be imparted to the ring. This can be effected by looping a string to a peg at E, winding up the string and pulling it out sharply.

This being done, the point O may be held by a string, and OC placed at any inclination to the vertical.

Or, if a small peg be fixed at O underneath the rim of OC, this may be placed on the cup K of a fixed vertical stand KL, and then the axis OC may be seen to revolve horizontally.

244. *Steady motion, about a fixed point O, of a rigid body so constituted that two of the principal moments of inertia at the point O are equal.*

In this case, of which the preceding is a particular case, the constraining couple is equal to

$$(C\Omega n - A\Omega^2 \cos\alpha)\sin\alpha.$$

If $Cn = A\Omega\cos\alpha$, this expression vanishes, and the steady motion continues without the action of any extraneous force.

245. *Motion of a rigid body about a fixed point O under the action of no extraneous forces, the body being so constituted that A and B are equal at the point O.*

The body is supposed to be set in motion in a given manner, or by means of given impulses.

The moments about the principal axes of the given impulses are the initial values of the angular momenta

$$A\omega_1,\ A\omega_2,\ C\omega_3.$$

There being no forces in action the angular momenta about any fixed lines through the point O remain unchanged and therefore the resultant angular momentum is a constant quantity, H, and the axis of resultant angular momentum is a fixed line.

This line, which is called the invariable line, we shall take for Oz.

Take OC as the axis of (3), and, for axes of (1) and (2) the line in the plane zOC perpendicular to OC, and the line perpendicular to the plane zOC.

Then if ψ be the inclination of the plane zOC to a fixed plane through Oz,

$$\theta_1 = -\dot{\psi}\sin\theta,\ \ \theta_2 = \dot{\theta},\ \ \theta_3 = \dot{\psi}\cos\theta,$$

and the equations of motion are

$$A\dot{\omega}_1 - A\omega_2\dot{\psi}\cos\theta + C\omega_3\dot{\theta} = 0, \text{.................(i)}$$

$$A\dot{\omega}_2 + C\omega_3\dot{\psi}\sin\theta + A\omega_1\dot{\psi}\cos\theta = 0, \text{.........(ii)}$$

$$C\dot{\omega}_3 - A\omega_1\dot{\theta} - A\omega_2\dot{\psi}\sin\theta = 0. \text{...............(iii).}$$

We have also $\omega_1 = -\dot{\psi}\sin\theta$, and $\omega_2 = \dot{\theta}$, and therefore, from (iii), it follows that ω_3 is constant.

Now, $H\cos\theta = C\omega_3$, and therefore it follows that θ is constant.

From equation (i), we find that ω_1 is constant and therefore that $\dot{\psi}$ is constant.

Further since $-H\sin\theta = A\omega_1$, we obtain $A\dot{\psi} = H$.

The condition, $C\omega_3 \sin\theta + A\omega_1 \cos\theta = 0$, derived from (ii), simply expresses the known fact that the angular momentum about the line in the plane zOC which is perpendicular to Oz is zero.

The conclusion is that the angular velocity about OC is constant, and that the plane zOC revolves with uniform angular velocity H/A about the axis of resultant angular momentum Oz.

246. What has been proved of motion about a fixed point is equally true of the motion of a rigid body relative to its centre of gravity.

This can be easily illustrated by tossing into the air a solid body of any symmetrical shape, such as a piece of wood in the form of a circular cylinder, or in the form of a circular cone, or any regular prism or regular pyramid, taking care to give the body rather a rapid rotation about its axis.

In all such cases, in whatever manner the body may be thrown up, its axis will be seen to describe uniformly a right circular cone about a line through the centre of gravity the direction of which remains unchanged.

This line is the axis of the resultant angular momentum originally imparted to the body.

If the Gyroscope described in Art. (243) be mounted in a fixed framework so as to give the diameter OC free motion about the fixed point G, we obtain an apparatus for directly demonstrating the fact that the earth has a motion of rotation independently of its motion of translation.

The ring being set rotating with great rapidity, so as to continue rotating for some hours, it will be seen that its position relative to the room in which the machine is situated gradually changes; and, as we know that the direction of the axis of rotation of the ring does not change, it follows that the earth itself is in a state of rotation.

247. *Euler's Equations.* In the case of the motion of a single rigid body about a fixed point or about its centre of

gravity, if we take for our moving axes three straight lines which are always coincident with three given lines in the body,

$$\theta_1 = \omega_1,\ \theta_2 = \omega_2,\ \theta_3 = \omega_3.$$

If further we take these three lines to be principal axes,

$$h_1 = A\omega_1,\ h_2 = B\omega_2,\ h_3 = C\omega_3,$$

and the equations of motion take the forms

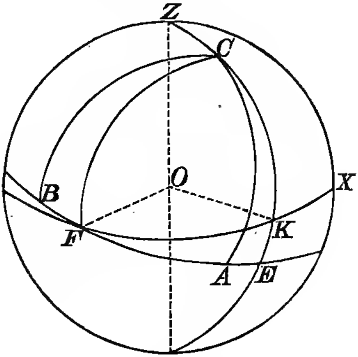
$$A\dot{\omega}_1 - (B - C)\,\omega_2\omega_3 = L,$$
$$B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 = M,$$
$$C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 = N,$$

which are Euler's Equations.

Geometrical equations connecting the angular motions about three moving axes with the motion of the body referred to lines fixed in space.

If OA, OB, OC be three lines fixed in the body, and OX, OY, OZ three lines fixed in space, and if these lines end on the surface of a sphere, the position of the body will be completely determined by the quantities ψ, θ, ϕ, if

$$\psi = CZX,\ \theta = ZC,\ \phi = ACE.$$



Moreover the motion is completely determined by the rotation $\dot{\theta}$ about OF, the rotation $\dot{\psi}$ about OZ and by the rate of separation $\dot{\phi}$ of the plane OCA in the body from the moving plane ZCE.

This system must be equivalent to the system of angular rotations ω_1, ω_2, and ω_3 about OA, OB and OC.

Expressing this equivalence we obtain,

$$\omega_1 = \dot{\theta} \cos FA + \dot{\psi} \cos ZA = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi,$$

$$\omega_2 = \dot{\theta} \cos FB + \dot{\psi} \cos ZB = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi,$$

$$\omega_3 = \dot{\psi} \cos \theta + \dot{\phi}.$$

These equations may be employed in the discussion of the general problem of the motion of a rigid body about a fixed point.

For this however we refer the student to Poinsot's *Nouvelle Théorie de la Rotation des corps solides,* to Dr Routh's *Rigid Dynamics*, and other treatises.

248. *Impulses.* We have given in Arts. (200) and (201), the principles which determine the effects of impulses, and the equations for the calculation of those effects.

If smooth inelastic bodies impinge on each other, it must be carefully borne in mind, that the velocities, immediately after impact, of the points of the bodies in contact with each other, are the same in the direction of the common normal to their surfaces.

If however two perfectly rough inelastic bodies impinge on each other, the geometrical condition is that the velocities of the points of contact are the same in any direction.

If when a system is in motion, a straight line in the system is suddenly fixed, the impulses called into action have no moment about the line, and consequently the angular momentum about it remains unchanged.

If a point of the system be suddenly fixed, the change of motion is determined by the fact that the angular momentum round any straight line whatever, through the point, remains unchanged.

In the case of a single rigid body, if u, v, w be the velocities of the centre of gravity, and ω_1, ω_2, ω_3 the angular velocities about principal axes through the centre of gravity, the changes of these quantities due to impulses whose components are P, Q, R, and moments, about axes through the centre of gravity, G, H, K are given by the equations

$$M(u'-u)=P,\ M(v'-v)=Q,\ M(w'-w)=R,$$
$$A(\omega_1'-\omega_1)=G,\ B(\omega_2'-\omega_2)=H,\ C(\omega_3'-\omega_3)=K.$$

The solution of problems involving impulses may sometimes be facilitated by the use of the principle of virtual velocities.

For any imagined geometrical displacement the virtual moment of the changes of linear and angular momenta is equal to the virtual moment of the applied impulses.

It may be well to notice that if a couple be displaced about a line parallel to its plane the virtual work is zero, so that if a couple is displaced about an axis not perpendicular to its plane, all that is necessary is to find the component of the couple about the axis of displacement.

The solutions of the two following problems will serve as further illustrations of the application of the principles of momentum and energy.

249. *A system, consisting of four equal rods forming a square ABCD, having universal joints at A, B, C and D, is rotating freely with an angular velocity n about the line EF joining the middle points of BC and DA; it is required to determine the changes of motion produced when the point A is suddenly fixed.*

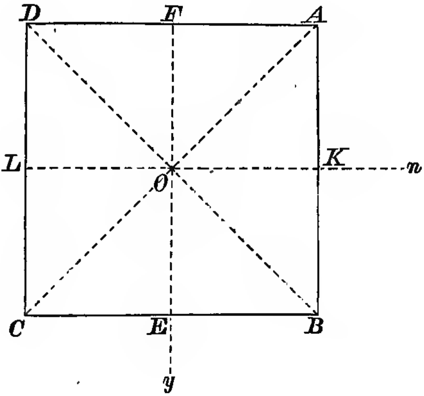
In order to mark directions take the axis of z perpendicular to the plane of the square, and let ω_1, ω_2, ω_3, ω_4 be the angular velocities of AB, BC, CD, DA immediately after A is fixed.

Expressing the fact that the angular momentum of the rods AB, BC about the straight line AC is unchanged, we obtain

$$\tfrac{4}{3}\frac{a^2\omega_1}{\sqrt{2}} + a(\omega_2 + 2\omega_1)\frac{a}{\sqrt{2}} - \frac{a^2\omega_2}{3\sqrt{2}} = -\frac{a^2 n}{\sqrt{2}} - \frac{a^2 n}{3\sqrt{2}},$$

or

$$5\omega_1 + \omega_2 = -2n.$$



Further the angular momenta of ADC about AC, and of BCD about BD are unchanged, and therefore

$$\tfrac{4}{3}\frac{a^2\omega_4}{\sqrt{2}} + a(\omega_3 + 2\omega_4)\frac{a}{\sqrt{2}} - \frac{a^2\omega_3}{3\sqrt{2}} = \frac{a^2 n}{3\sqrt{2}} + \frac{a^2 n}{\sqrt{2}},$$

$$a(\omega_2 + 2\omega_1)\frac{a}{\sqrt{2}} + \frac{a^2\omega_2}{3\sqrt{2}} + a(\omega_3 + 2\omega_4)\frac{a}{\sqrt{2}} + \frac{a^2\omega_3}{3\sqrt{2}} = \frac{a^2 n}{3\sqrt{2}} + \frac{a^2 n}{\sqrt{2}},$$

or

$$5\omega_4 + \omega_3 = 2n,$$

$$3\omega_1 + 2\omega_2 + 2\omega_3 + 3\omega_4 = 2n.$$

Further we have the geometrical condition obtained by equating the two expressions for the velocity of C which is

$$2a\omega_2 + 2a\omega_1 = 2a\omega_3 + 2a\omega_4$$

or

$$\omega_1 + \omega_2 = \omega_3 + \omega_4.$$

From these equations we find that

$$-\frac{\omega_1}{9} = \frac{\omega_2}{17} = \frac{\omega_3}{3} = \frac{\omega_4}{5} = \frac{n}{14}.$$

If P, Q, R be the impulses at B, C, D, we obtain, by taking moments about A, B, C, for the rods AB, BC, CD,

$$k^2\omega_1 + a^3\omega_1 + a^2 n = 2aP,$$

$$a^2(\omega_2 + 2\omega_1) + k^2(\omega_2 - n) = 2aQ,$$

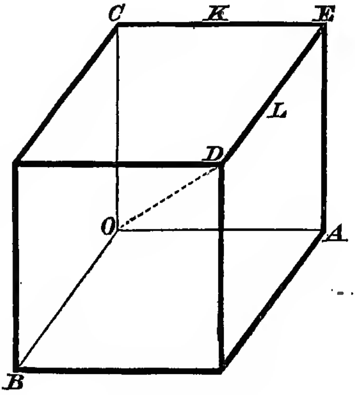
$$a^2(\omega_3 + 2\omega_4) - a^2 n - k^2\omega_3 = 2aR,$$

and therefore $Q = 0$, $14P = na$, $14R = -na$.

Finally the impulse at A which is the change of linear momentum of the system

$$= a\omega_1 + (a\omega_2 + 2a\omega_1) + (a\omega_3 + 2a\omega_4) + a\omega_4 = \tfrac{4}{7}na.$$

250. *A cube, the edges of which are twelve equal uniform rods hinged together, is hung up by one corner, the cube form being maintained by a string joining this corner with the lowest corner. It is required to find the initial change of stress at the point of support when the string is cut.*



The corner O being the point of support and the diagonal

OD vertical, it is clear that the initial angular accelerations of OA, OB, OC will be respectively in the planes AOD, BOD, COD and will be equal to each other; and further that the angular accelerations of all the other rods will be the same and will be, respectively, in parallel planes.

If ω represent this initial angular acceleration, $2a\omega$, which we shall call $2f$, will be the linear acceleration of A in the direction AD, of B in direction BD, and of C in direction CD.

Taking accelerations parallel to OA, OB, and OC, we obtain the following forms, where K and L are the centres of the rods CE, ED.

The accelerations of C are

$$f\sqrt{2},\ f\sqrt{2},\ 0,$$

the accelerations of K relative to C are

$$0,\ f/\sqrt{2},\ f/\sqrt{2},$$

of E relative to C,

$$0,\ f\sqrt{2},\ f\sqrt{2},$$

and of L relative to E,

$$f/\sqrt{2},\ 0,\ f/\sqrt{2}.$$

Therefore the actual accelerations of K are

$$f\sqrt{2},\ 3f/\sqrt{2},\ f/\sqrt{2},$$

and of L

$$3f/\sqrt{2},\ 2f\sqrt{2},\ 3f/\sqrt{2}.$$

Now suppose a displacement made by slightly increasing the length of OD, so as to turn every rod through a small angle θ.

The displacements of the various points follow the law of the accelerations and are of the same forms, replacing ω by θ.

Observing that there are six rods under the same conditions as CE, three under the same conditions as ED, and three other rods OA, OB, OC, the equation of virtual work is

$$6mfa\theta\left(2+\tfrac{9}{2}+\tfrac{1}{2}\right)+3mfa\theta\left(\tfrac{9}{2}+8+\tfrac{9}{2}\right)$$
$$+3m\frac{4a^2}{3}\omega\theta+9m\frac{a^2}{3}\omega\theta=12mga\theta\sqrt{6},$$

for the displacement of D is the resultant of the displacements of A, B, and C.

From this equation we obtain

$$25a\omega = 3g\sqrt{6},$$

and since the acceleration of D is $2a\omega\sqrt{6}$, it follows that the acceleration of G is $a\omega\sqrt{6}$ and is therefore $18g/25$.

Hence it follows that the diminution of stress at O is 18/25 of the total weight of the system.

The initial stresses at the several joints can be obtained by giving independent displacements to the several rods, breaking the connections at the different joints.

251. The endeavour of this chapter has been to explain the use of moving axes, and to illustrate the solution of problems in three dimensions by the direct application of fundamental conceptions and of equations derived immediately from these conceptions.

It is hoped that these illustrations will serve as an introduction to a very extensive field of thought, a field which has been worked extensively by many mathematicians with great power and success.

For the more general problems of the motions of solid bodies on surfaces, for the great problems of Precession and Nutation, and for the utilization and application of systems of generalized coordinates, the student will consult such English works as the *Natural Philosophy* of Sir William Thomson and Professor Tait, Dr Routh's *Rigid Dynamics*, and the *Treatise on Generalised Coordinates* by Dr H. W. Watson and Mr S. H. Burbury.

The student who has time at his disposal will study the great works of Laplace and Lagrange, and those works, with the English Treatises to which we have referred, and the treatises and papers of Jacobi, Laurent, Bertrand, Resal, Mathieu and numerous others, offer a vast area for consideration, and suggest unlimited fields of research.

EXAMPLES.

1. Two equal smooth circles are fixed so as to touch the same horizontal plane, their planes being at different inclinations; two small heavy beads are projected at the same instant along these circles from their lowest points, the velocity of each bead being that due to the height of the highest point of the other circle above the horizontal plane, shew that during the motion the two beads will always be at equal heights above the horizontal plane.

2. A sphere is projected horizontally on an inclined plane, the surface of which is perfectly rough; shew that its centre will describe a parabola.

3. An imperfectly elastic rough sphere is projected obliquely without rotation against a fixed plane; if i, i', be the angles of incidence and reflection, λ the coefficient of elasticity for direct impact, and ρ the ratio of the tangential forces of restitution and compression, prove that

$$2\rho = 5 - 7\lambda \tan i' \cot i.$$

4. Two particles of masses m, $2m$ are fixed to the ends of a weightless rod of length $2a$ which is freely moveable about its middle point. Prove that if θ be the inclination of the rod to the vertical when the particles are moving with uniform angular velocity ω,

$$3\omega^2 a \cos\theta = g.$$

5. A solid rectangular parallelepiped with edges of length a, b, c, is acted on by instantaneous couples with axes parallel to these edges and of moments proportional to $p : q : r$; shew that the direction cosines of the instantaneous axis of rotation are in the ratio

$$\frac{p}{b^2 + c^2} : \frac{q}{c^2 + a^2} : \frac{r}{a^2 + b^2}.$$

6. If an octant of an ellipsoid bounded by three principal planes be rotating about the axis a with angular velocity ω, and if this axis suddenly become free, and the axis b fixed, shew that the new angular velocity is $\dfrac{2ab\omega}{\pi(a^2+c^2)}$.

7. A ring, mass M, is placed on a smooth table, and is held fast while a circular disc of smaller radius and mass m, is made to run round the inside of it, the coefficient of friction being infinite. The ring being now set free determine completely the motion and the paths traced out by the centres of the ring and disc respectively.

8. There are n discs placed on a smooth table, with their edges in contact, and their centres in a straight line, the edges of the discs being perfectly rough. The first disc has communicated to it a velocity u in a direction at right angles to the line of centres. Prove that the velocity and angular velocity of the pth disc are given by

$$\left.\begin{matrix} u_p \\ \\ \kappa\omega_p \end{matrix}\right| = C\left\{(\kappa+a)\left(\frac{\kappa+a}{\kappa-a}\right)^{n-p} \pm (\kappa-a)\left(\frac{\kappa-a}{\kappa+a}\right)\right\}u,$$

where
$$C = \frac{(\kappa^2-a^2)^{2n-1}}{(\kappa-a)^{2n-1}+(\kappa+a)^{2n-1}},$$

and a, k are the radius and radius of gyration of each disc.

9. Two equal rods AB, BC are jointed at one extremity B of each, and the other end A of one is fixed: if C be held in such a position that ABC is a right angle and AC horizontal, prove that when C is suddenly let go the initial pressure at B will be $\frac{1}{4}$ of the weight of either rod and horizontal.

10. A sphere is moving at a given moment on an imperfectly rough horizontal table with velocity V, and at the same time an angular velocity Ω round some horizontal diameter, the angle between the direction of V and the axis of Ω being α: prove that the centre will describe a parabola

if $$ak^2\Omega^2 + \Omega V \sin\alpha\,(a^2-k^2) - aV^2 = 0.$$

11. One end of a heavy rod rests on a horizontal plane and against the foot of a vertical wall, the other end rests against a parallel vertical wall, all the surfaces being smooth. Shew that if it slips down, the angle ϕ through which it turns round the common normal to the vertical walls is given by the equation,

$$\frac{d\phi^2}{dt^2}\left(1 + 3\cos^2\phi\right) = -\frac{6g}{\sqrt{(a^2 - b^2)}}\sin\phi + C,$$

where $2a$ is the length of the rod, and $2b$ the distance between the walls.

12. A smooth plate inclined at an angle ϕ to the horizon is made to rotate about a vertical axis AB with uniform angular velocity ω. A rod of mass m is compelled by guides to be always vertical, and at a distance r from AB, while it rests with one end in contact with the plate, sliding up and down as the latter rotates. Shew that, if the rod be initially in its lowest position, the pressure on the plate at the end of the time t will be

$$m\,(g\cos\phi + r\omega^2\cos\omega t\sin\phi)\sec^2\phi.$$

13. A square is revolving, with angular velocity Ω, round a diagonal which is vertical, the highest point of which is fixed; if suddenly one of the angles adjacent to the fixed point become fixed, shew that the square will just revolve round the fixed side, if $\Omega^2 = \dfrac{3\sqrt{2}g}{a}$, and the impulsive pressures at the two fixed points are in the ratio 3 : 2; a being the length of a side of the square.

14. A smooth homogeneous cone is placed within a hollow sphere circumscribing it, and set rotating about its axis of figure with a given angular velocity; determine the motion.

15. A plane mirror whose thickness may be neglected and whose centre of gravity is fixed, moves under the action of no forces so that its invariable plane is horizontal; prove that the image of any luminous point below the mirror and

on the invariable line moves on the surface of a sphere so that its zenith distance (θ) and azimuth (ψ) are connected by the equation

$$\frac{d\psi}{dt} = m + n \operatorname{cosec}^2 \frac{\theta}{2}.$$

16. A sphere of radius c on a smooth horizontal plane is placed in contact with a rough vertical plane, which is made to revolve with angular velocity ω about a vertical axis in itself: if a be the initial distance of the point of contact from the axis, and r the distance at the time t, prove that

$$2r = \left(a + c\sqrt{\tfrac{7}{5}}\right) \epsilon^{\sqrt{\frac{5}{7}}\omega t} + \left(a - c\sqrt{\tfrac{7}{5}}\right) \epsilon^{-\sqrt{\frac{5}{7}}\omega t}.$$

Also shew that the ratio of the friction to the pressure approximates, as t increases indefinitely to $1 : \sqrt{35}$.

17. A rough plane is made to revolve uniformly, with angular velocity ω, about a horizontal line in itself, and a sphere is projected so as to move upon it, determine the motion; and if, when the plane is horizontal, the centre of the sphere be vertically above the axis of revolution, and be moving parallel to it, prove that the contact will cease when the plane has revolved through an angle θ given by the equation

$$\frac{11 \cos\theta}{6} = \frac{a\omega^2}{g} + \frac{5}{12}\left(\epsilon^{\theta\sqrt{\frac{5}{7}}} + \epsilon^{-\theta\sqrt{\frac{5}{7}}}\right).$$

18. A frame consists of four equal uniform rods loosely jointed at their ends so as to form a square, and one of the rods carries a light ring fastened to it at its middle point. The frame moves with uniform velocity on a table. All kinds of friction being neglected, prove that when a vertical bolt is shot through the ring the frame will be brought absolutely to rest.

19. A solid cube is in motion about any angular point which is fixed; if ω_1, ω_2, ω_3 be the angular velocities about the three edges through the fixed angular point, and if no extraneous forces be in action, prove that $\omega_1 + \omega_2 + \omega_3$ is constant, and that $\omega_1^2 + \omega_2^2 + \omega_3^2$ is also constant.

20. A circular wire ring of radius a rolls on a rough horizontal plane, so that its plane maintains a constant inclination (α) to the vertical; if ω be the angular velocity of the ring, and Ω the azimuthal motion of its centre, prove that

$$4a\omega\Omega\cos\alpha - a\Omega^2\sin\alpha\cos\alpha = 2g\sin\alpha.$$

21. A hollow cone, the internal surface of which is perfectly rough, is fixed in a position in which its axis is inclined at an angle α to the vertical, and a solid cone of smaller vertical angle is placed inside, its vertex coinciding with the vertex of the fixed cone, and allowed to perform small oscillations; prove that the length of the simple isochronous pendulum is

$$\frac{4k^2}{3h}\frac{\sin(\beta-\gamma)}{\sin\alpha\sin^2\gamma},$$

2α and 2γ being the vertical angles, h the height of the moving cone, and k its radius of gyration about a generating line.

22. A vertical hollow infinitely rough cylinder is moveable about its axis. A sphere is projected horizontally in contact with the cylinder. Shew that the cylinder will move during the subsequent motion with a constant angular velocity, and find its magnitude, having given V the velocity of projection of the sphere before it touched the surface, a, m the radius and mass of the sphere, and b, M those of the cylinder.

23. A sphere is pressed between two perfectly rough parallel boards which are made to revolve with the uniform angular velocities Ω and Ω' about fixed axes A and B perpendicular to their planes. Prove that the centre of the sphere describes a circle about an axis which is in the same plane as A and B, and whose distances from these axes are inversely proportional to Ω and Ω'.

Shew that, when A and B coincide, the points of contact will trace on the sphere small circles, the tangents of whose radii will be $\frac{c}{a}\frac{\Omega \sim \Omega'}{\Omega + \Omega'}$, a being the radius of the sphere, and c that of the circle described by its centre.

24. If a rectangular parallelopiped (edges $2a$, $2a$, $2b$) move freely about its centre of gravity under no forces: shew that its angular velocity about one principal axis is constant and about the other principal axes is periodic, the period being to the period of revolution about the first-mentioned principal axis as $b^2 + a^2 : b^2 - a^2$.

25. A system of two bodies is acted on by no forces, and moving, one about a fixed axis, the other about an axis fixed in the first, and parallel to the former axis; shew under what circumstances the two bodies will revolve about the fixed axis as a single body, and determine when disturbed from such a condition, whether they will oscillate or not, and the time of a small oscillation. Prove that if the bodies be two equal rods freely jointed at a common extremity and moveable about the other extremity of one of them, the time of a small oscillation $= \sqrt{\frac{7}{48}} \times$ time of a complete revolution.

26. A body in the form of a solid of revolution has a point in its axis attached by a universal joint and a string of length l to a fixed point; an angular velocity ω about its axis of figure is given to the body, and n is the angular velocity of the centre of gravity about the vertical, θ the angle which the string, ϕ that which the axis of the body makes with the vertical, a the distance between the centre of gravity and the point of suspension; C being the moment of inertia about the axis of figure. Shew that when the motion is steady

$$(A - C)\, n \sin\phi\, (\omega - n\cos\phi) \cos\theta = Mga \sin(\phi - \theta)$$

$$\omega^2 (l \sin\theta + a \sin\phi) = g \tan\theta.$$

27. A ring rests upon two smooth horizontal bars which in the position of equilibrium subtend an angle 2α at the centre; shew that, if the ring be disturbed by twisting it through a small angle about its vertical diameter, the length of the simple isochronous pendulum will be $\frac{c}{2} \cot\alpha \operatorname{cosec}\alpha$.

28. A smooth circular tube, of radius a, has its centre fixed at a height h above a fixed horizontal plane, which it touches, a particle of mass m is placed within it at a distance c below the horizontal plane through the centre of the tube which becomes z at the time t, prove that, if the moment of inertia of the tube about a diameter be nma^2

$$\left(\frac{dz}{dt}\right)^2 = \frac{2g}{a^2} \cdot \frac{(z-c)(h^2-z^2)(nh^2+a^2-z^2)}{(n+1)h^2-z^2}.$$

If the particle be initially in the horizontal diameter, prove that the initial pressure between the tube and particle will be less than if the ring were fixed, in the ratio $n : n+1$.

29. A sphere is rolling on the rough surface of a cylinder, the cross section of which is the curve

$$3r = a\sqrt{2}\,\epsilon^{\theta\sqrt{\frac{2}{7}}};$$

prove that, if there be no forces, the path of the point of contact becomes, when the cylinder is developed into a plane, a curve of the form,

$$y = (\alpha + \beta x)\cos\left(\log\frac{x}{c}\right) + (\gamma + \delta x)\sin\left(\log\frac{x}{c}\right).$$

30. A sphere moves under the action of gravity on the inside of a rough cylindrical surface, of which the generating lines are inclined at an angle α to the horizon, and the transverse section perpendicular to the generating lines is a cycloid with its vertex at the lowest generating line.

The sphere is projected initially with a velocity V along the generating line at which the curvatures of the sphere and cycloid are equal.

Prove that the motion will be comprised within a length $14V\sqrt{\dfrac{2a}{5g\cos\alpha}}$ of the cylinder, and that the time between successive instants of the sphere reaching the original generating line is $4\pi\sqrt{\dfrac{7a}{5g\cos\alpha}}$, where a is the radius of the generating circle of the cycloid.

31. A perfectly rough circular disc, of which the centre is fixed, is constrained to roll with its edge on a horizontal plane, so that the point of contact moves with angular velocity n.

A sphere is placed on the disc in contact with a horizontal diameter, and in a state of relative rest. Obtain the equations of motion, and shew that the sphere moves *down* the disc as if that were at rest and the sphere a smooth heavy particle, and that the trace of the point of contact on the disc, referred to axes in its plane, that of y being horizontal, is the curve

$$y = \frac{c}{2}\left\{e^{n\sqrt{\frac{2x\cos\alpha}{g}}} + e^{-n\sqrt{\frac{2x\cos\alpha}{g}}}\right\};$$

the disc being inclined at an angle α to the vertical, and c being the initial value of y.

32. At a point P of the earth's surface a sphere has its centre O fixed and by means of pegs at the extremity C of a diameter, the diameter OC is compelled to move in the meridian at P; if originally OC be parallel to the axis of the earth and the sphere have angular velocity about OC; prove that if the sphere be disturbed OC will oscillate in the meridian in a time

$$\frac{2\pi}{\sqrt{n\omega}}$$

where ω is the angular velocity of the earth about its axis.

33. A solid of revolution has a point O in the circumference of its base fixed and receives a blow in a direction passing through G its centre of gravity and parallel to the tangent line to the base at O. If the principal moments of inertia at G in directions perpendicular to the blow are proportional to those about parallel axes through O, prove that the instantaneous axis and OG are equally inclined to the plane of the base.

34. A heavy, uniform, and inextensible string is in equilibrium in the form of a horizontal ring on a smooth sphere; prove that, if it be cut at a point A, the initial change of tension at a point P will be to the weight of the string in the ratio

$$\cos h\,(\phi\cos\alpha) : 2\pi\cot\alpha\cos h\,(\pi\cos\alpha),$$

α being the angular distance of the string from the vertex of the sphere, and $\pi-\phi$ the angle subtended at the centre of the ring by the arc PA. *Tripos, Jan.* 1883.

35. A rough heavy sphere, radius c, rolls on a fixed rough surface, of the form generated by the revolution of a circle, radius b, about a vertical axis in its own plane, distant a from the centre, a being greater than $b+c$. Prove that if, at the time t, ϕ be the angle through which the plane through the vertical axis and centre of the sphere has turned, θ the inclination to the vertical of the common normal, ω_3 the angular velocity about that common normal, α and λ the initial values of θ and $\dot\phi$, and if $\dot\theta$ and ω_3 be initially zero, and $l=b+c$,

$$c\dot\omega_3 = a\dot\theta\dot\phi,$$

$$l^2\dot\theta^2 + (a-l\sin\theta)^2\,\dot\phi^2 - \lambda^2\,(a-l\sin\alpha)^2 + \tfrac{2}{7}c^2{\omega_3}^2$$
$$= \tfrac{10}{7}gl\,(\cos\alpha-\cos\theta),$$

$$\frac{d}{d\theta}\left\{(a-l\sin\theta)^2\frac{d\omega_3}{d\theta}\right\} + \tfrac{2}{7}a\omega_3\,(a-l\sin\theta) = 0.$$

36. A perfectly rough plane, inclined at a fixed angle to the vertical, rotates about a vertical line with uniform angular velocity; shew that the path of a sphere which is placed upon it is given by two equations of the forms

$$\dot y + a\dot x + by = 0,\quad \ddot x - a\dot y + b'x = c,$$

the origin being the point where the vertical line meets the plane, and the axis of y being the straight line in the plane which is always horizontal.

37. Two equal spheres attracting each other, the force varying as the distance, are rolled upon a perfectly rough horizontal plane. Prove that they will describe ellipses about each other in the periodic time

$$2\pi\sqrt{\frac{7}{10\mu}}.$$

If the plane revolve with a uniform angular velocity ω about a vertical axis, prove that their centre of gravity will move in a circle with uniform angular velocity $\frac{2\omega}{7}$; and that their relative orbits will be such that each will appear to the other to describe a circle with uniform angular velocity

$$\sqrt{\frac{10\mu}{7}+\frac{\omega^2}{7^2}}\pm\frac{\omega}{7},$$

while the centre of that circle moves with uniform angular velocity

$$\sqrt{\frac{10\mu}{7}+\frac{\omega^2}{7^2}}\mp\frac{\omega}{7}$$

in another circle.

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